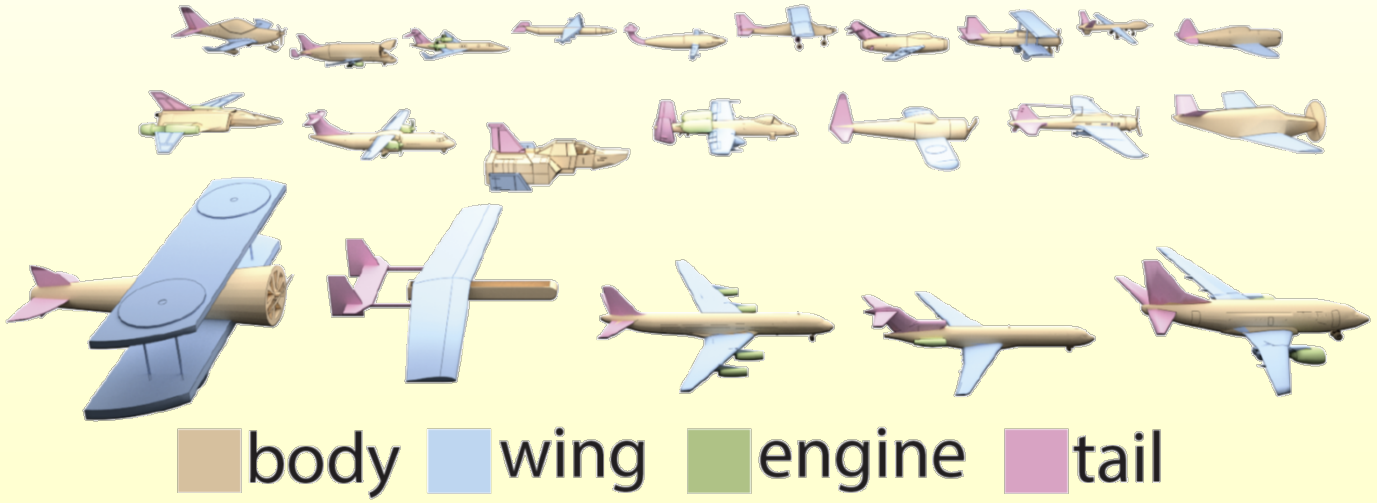
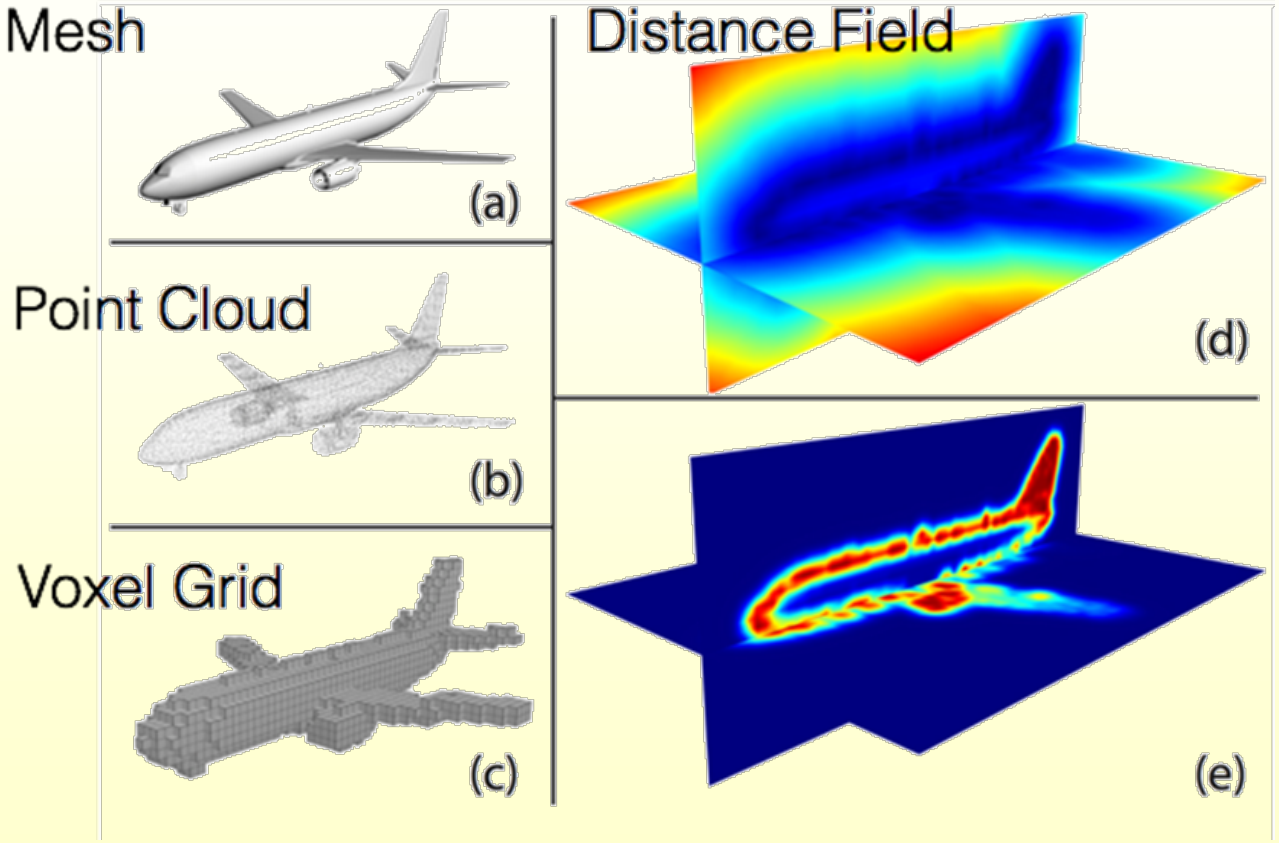
CS468:
Machine Learning for 3D Data

Anastasia Dubrovina, Leonidas Guibas, Hao Su
Stanford University

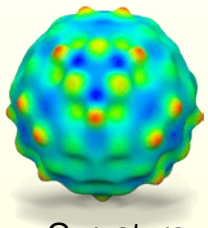


3D Representations



Functions on 3D Geometry

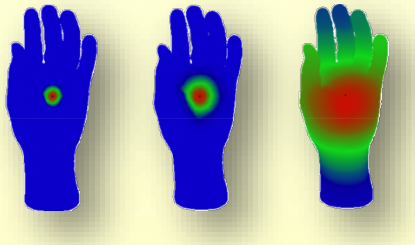
- Functions on data
  - Properties, attributes, descriptors, part indicators, etc.
  - But also beliefs, opinions, etc
- Operators on functions
  - Maps of functions to functions
    - Laplace-Beltrami operator on a manifold M



Curvature

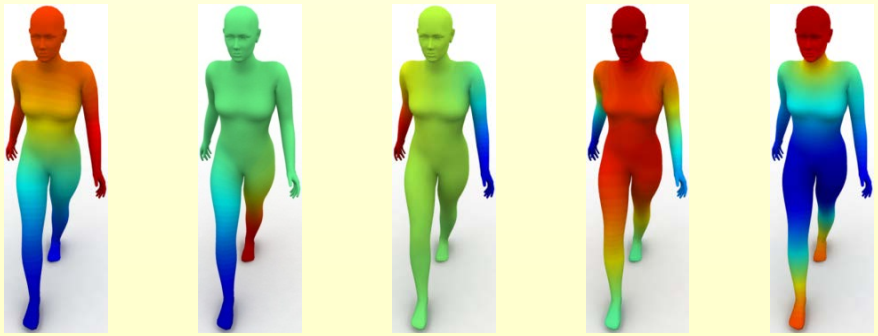
Parts

$$\Delta : C^\infty(M) \to C^\infty(M), \Delta f = \text{div } \nabla f$$



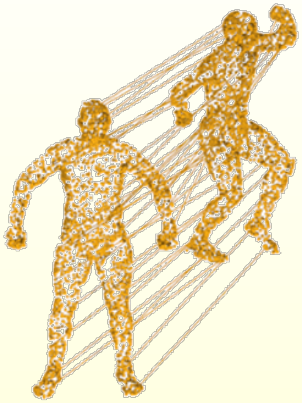
$$\frac{\partial u}{\partial t} = -\Delta u$$

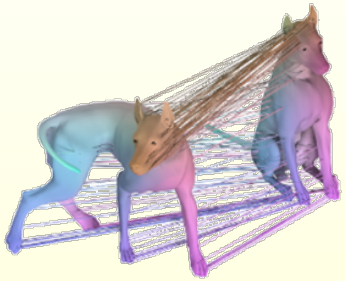
heat diffusion

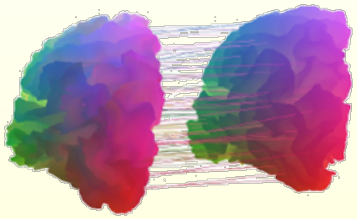


Laplace Beltrami eigenfunctions

Alignment and Correspondences, Shape Features



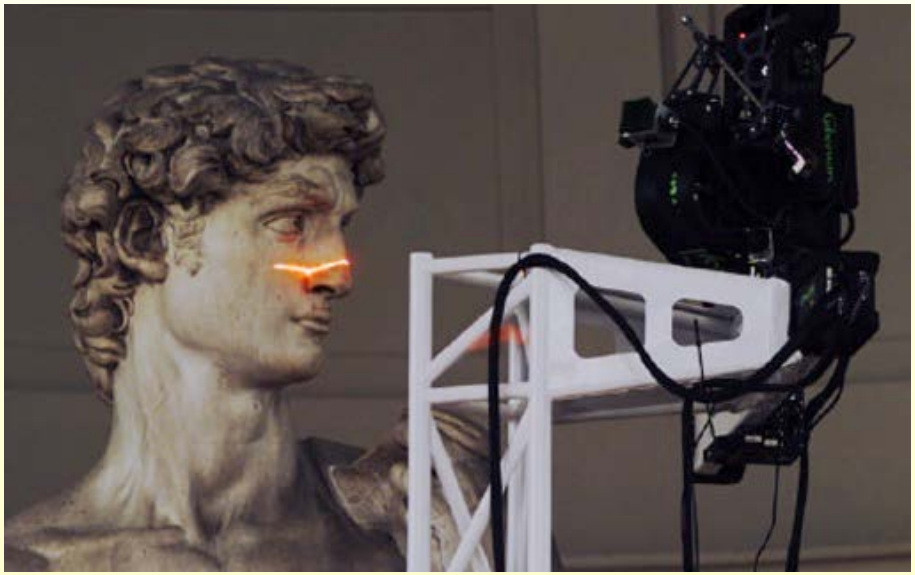




A B

Shape Acquisition – See CS348a

- Many 3D scanning techniques

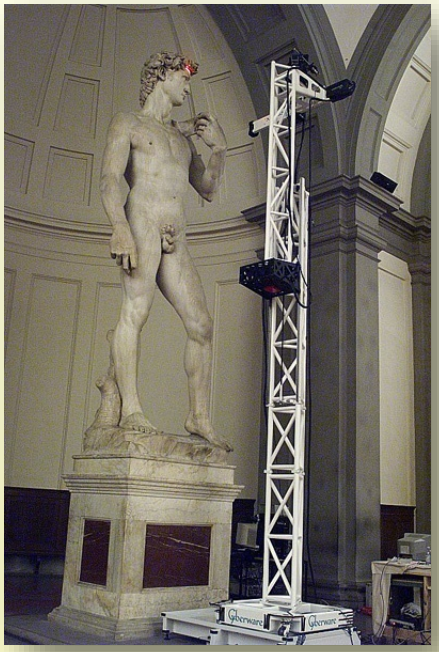


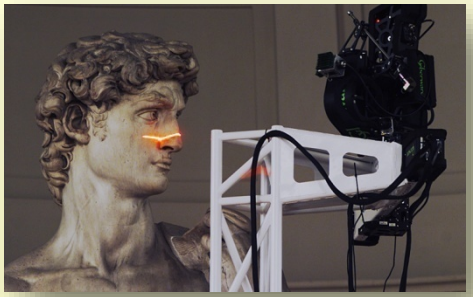
Triangulation scanner

Time of flight scanner

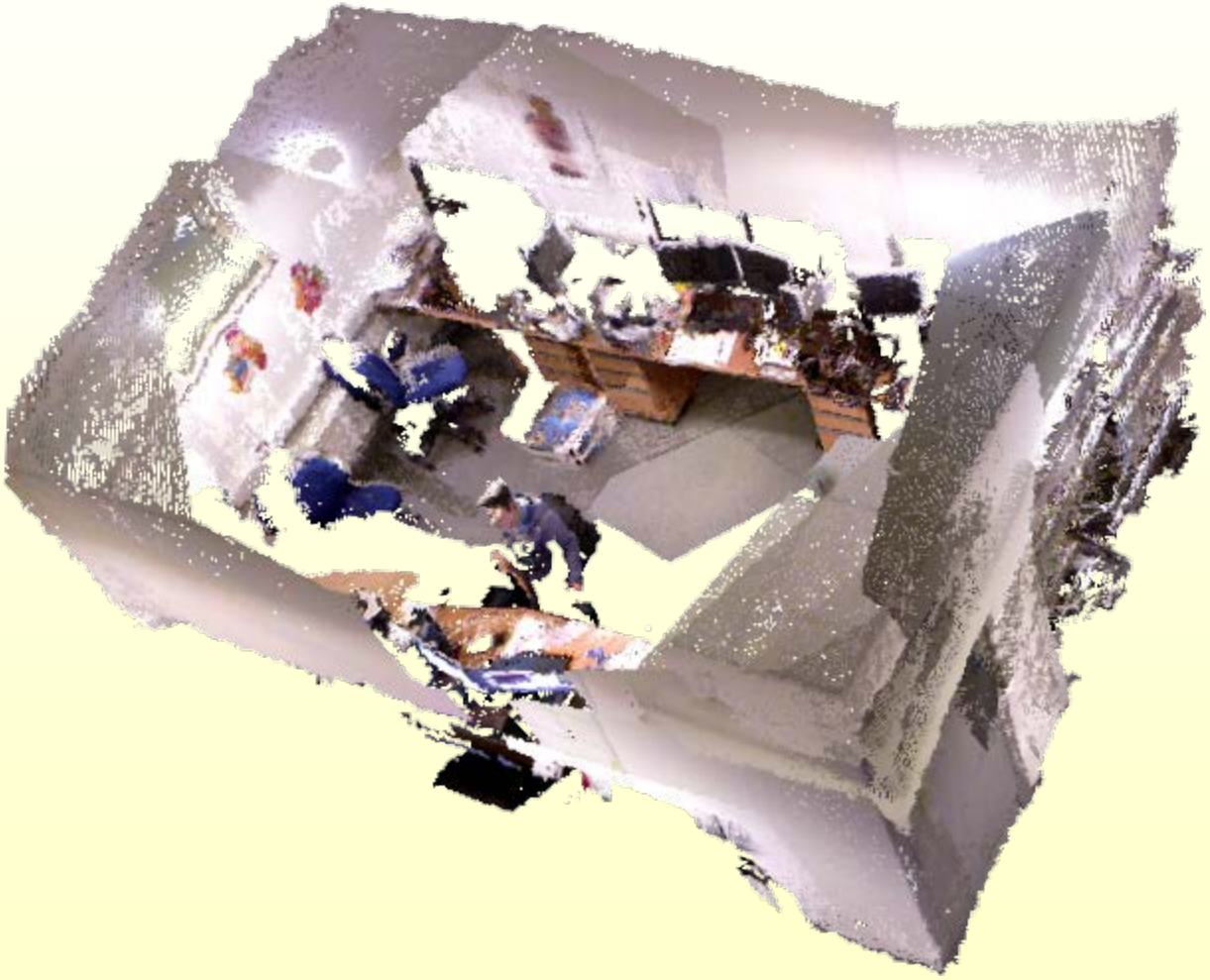
Aligning and Registering (Partial or Entire) Shapes

- The need to align and match shapes arises in many domains
- A classic example is shape acquisition through a 3D scanner





Simultaneous Localization and Mapping (SLAM)



Protein Structure Alignment

Human (red) and fly (yellow) thierodoxins, compared

The (3D) Registration Problem

Given:

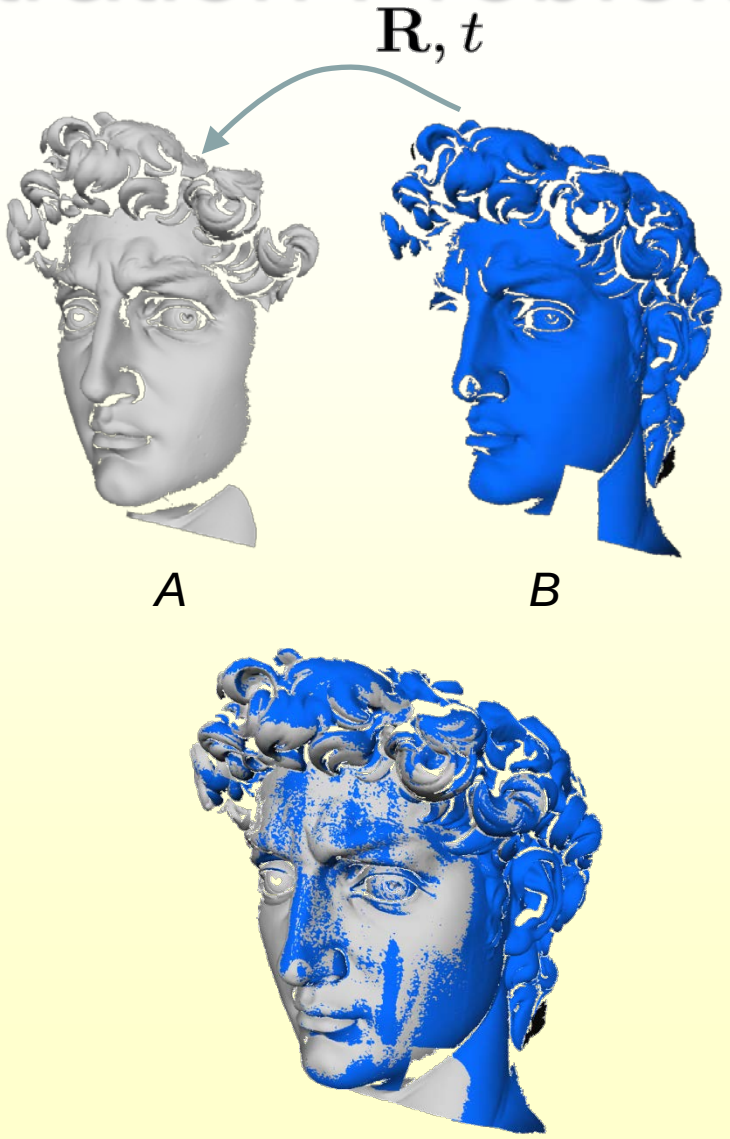
Two shapes *A* and *B* which partially overlap

Goal:

Using only rigid transforms, register *B* against *A* by minimizing a measure of distance between *A* and *B*

Assume:

A and *B* are positioned close to each other



Measuring Success: Shape Distances

Given two shapes *A* and *B*, we are interested in defining a distance or (dis-)similarity measure

$$\min_T \delta(A, T(B))$$ [extrinsic]

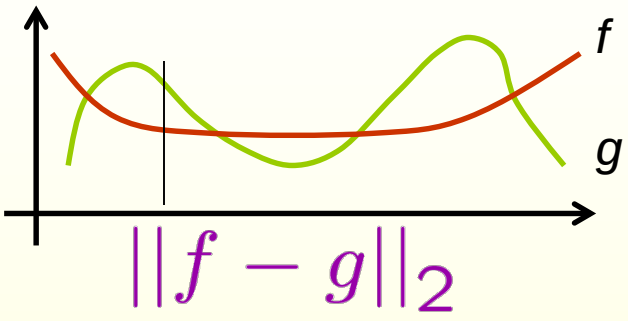
Such measures are crucial in shape similarity search, shape classification,and in general for defining ML loss functions.

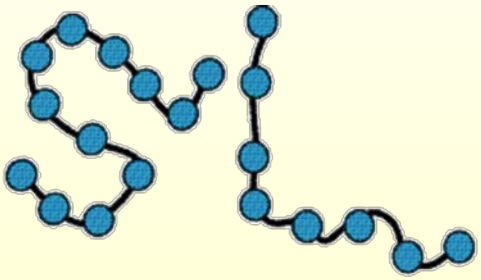
Issues about Distance Metrics

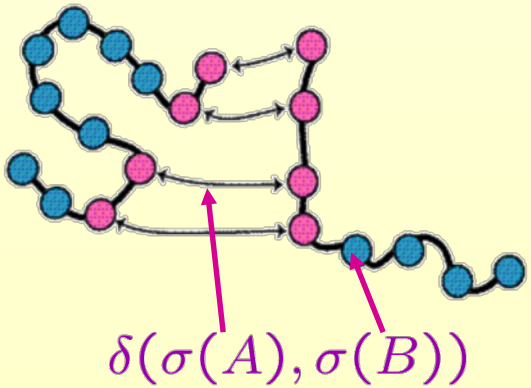
- We are all familiar with function norms (L_2, etc.). The common parametrization establishes correspondences. We don't have that for structures or shapes.
- Partial matches need to be considered -- notion of support σ for the match.
- What group of aligning transforms is to be considered?
- Is the resulting distance a metric?

Not for partial matches

$$\delta(A,C) \leq \delta(A,B) + \delta(B,C)$$

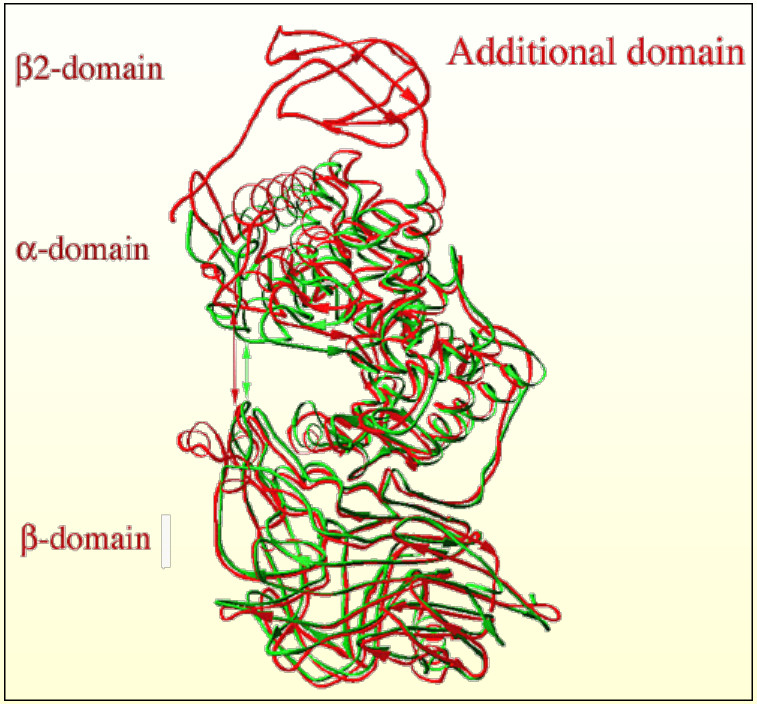


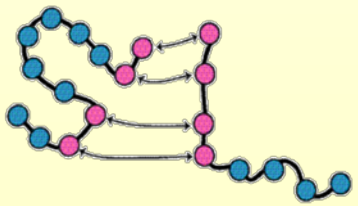




Simultaneous Estimation

- We are given two shapes *A* and *B*, each in its own coordinate system
- We must establish correspondences between certain parts (the alignment supports) of *A* and *B*
- We must find an optimal transform that best aligns the supports of *A* and *B*
- We must score this choice of supports and transform to produce a distance measure δ



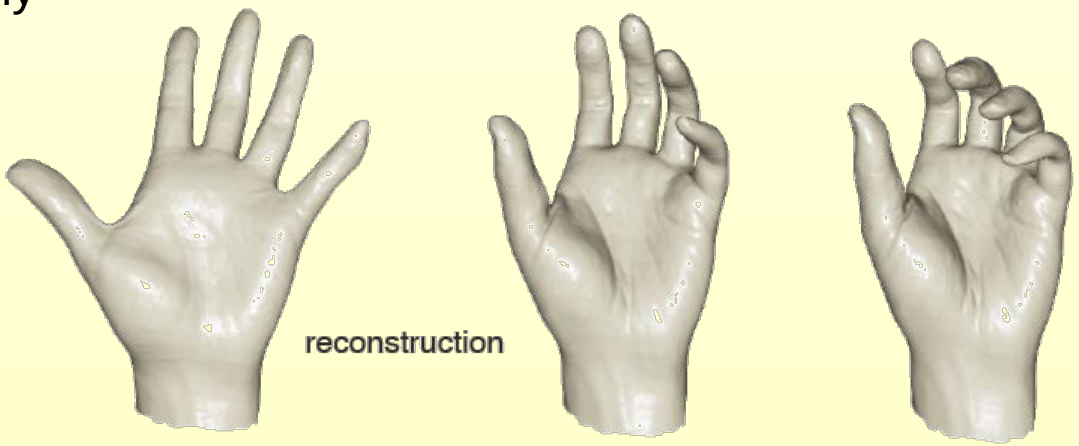


In computing the score,

1. what distance metric do we use?
2. how do we aggregate distances?
3. how do we trade-off larger supports for larger aggregate distance?

Degrees of Freedom

- Transform estimation
  - A rigid motion has 6 degrees of freedom (3 for translation and 3 for rotation)
  - We typically estimate the motion using many more pairs of corresponding points, so the problem is overdetermined (which is good, given noise, outliers, etc – use least squares approaches)
  - More general transforms require more degrees of freedom. When shape deformations are allowed, the degrees of freedom can grow very rapidly



A Double Whammy

- Estimate correspondences
- Estimate the aligning transform
- Gives rise to combinatorial searches
- Transforms can be non-linear

Hard optimization problems!

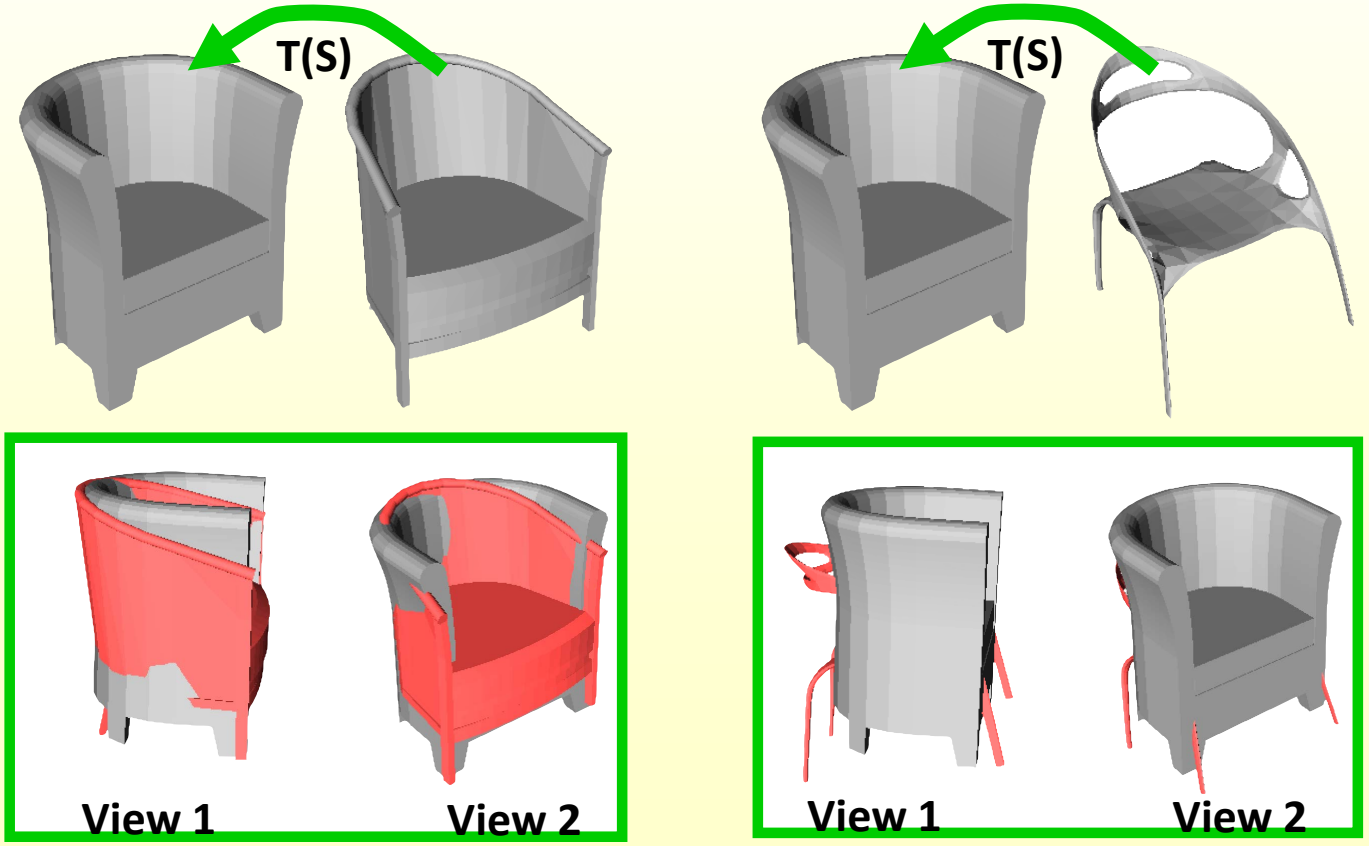
Good features help

Low-dimensionality of some transforms helps

Rigid Alignment

Rigid Alignment

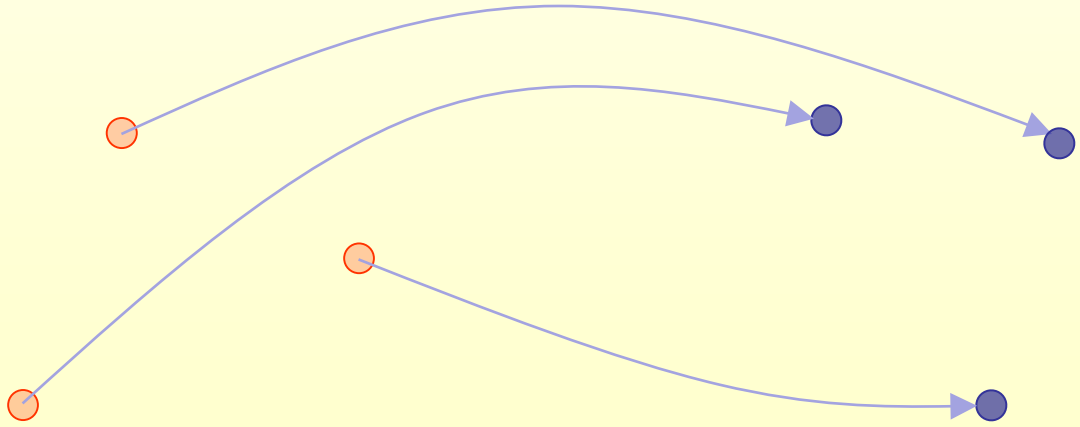
Search for a rigid motion that best aligns the shapes, even if the shapes are different



ICP: Optimal Transformation

Problem formulation, when given correspondences:

1. Given two sets points: $\{x_i\}, \{y_i\}, i = 1..n$ in $\mathbb{R}^3$? Find the rigid transform: $\mathbf{R}, t$ that minimizes: $\sum_{i=1}^{N} \|\mathbf{R}x_i + t - y_i\|_2^2$



Simplest Case: Rigid Alignment, Given Correspondences

- We are given two sets of corresponding points $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ in $\Re^3$. We wish to compute the rigid transform T that best aligns x_1 to y_1, x_2 to y_2, ..., and x_n to y_n.
- We define the error to be minimized by

$$\min_T \sum_{i=1}^{n} ||T(x_i) - y_i||^2$$

MSE error, RMS distance, …

- Old Problem:
  - Known and solved as the *orthogonal Procrustes problem* in Factor Analysis (Statistics) [Shönemann, 1966]
  - Known and solved as the *absolute orientation problem* in Photogrammetry [Horn, 1986]
  - Also
    mec
    stat
    etc
    ...

SVD-Based Solution

- A rigid motion T is a combination of a translation a and a rotation R, so that $T(x) = R(x) + a$.
- The quantity to be minimized is:

$$\min_{a,R} \left(\sum_{i=1}^{n} |R(x_i) + a - y_i|^2 \right)$$

The unknowns

SVD-Based Solution

- A rigid motion T is a combination of a translation a and a rotation R, so that $T(x) = R(x) + a$.
- If we place the origin of our coordinate system at the mean of the x_i's, then the quantity to be minimized simplifies to (up to some constants):

$$\min_{a,R} \left(\sum_{i=1}^{n} |y_i - a|^2 - 2 \sum_{i=1}^{n} \langle R(x_i), y_i \rangle \right)$$

T R

- Note that the translational and rotational parts factor. The translational part a can easily be seen to be optimized by

$$a = \frac{1}{n} \sum_{i=1}^{n} y_i$$

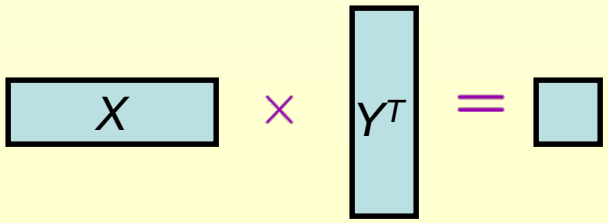
The centroids of the two point sets have to be aligned!

The Rotation Part

- Define

$$\begin{aligned} \overline{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ \overline{y} &= \frac{1}{n}\sum_{i=1}^{n} y_i \\ X &= [x_1 - \overline{x}, \ldots x_n - \overline{x}]^T \\ Y &= [y_1 - \overline{y}, \ldots y_n - \overline{y}]^T \end{aligned}$$

- Here *X* and *Y* are 3 by *n* matrices.



- Now compute the SVD*

$$XY^T = UDV^T \quad (3 \times 3)$$

- *U* and *V* are 3 by 3 orthogonal matrices, and *D* is a diagonal matrix with decreasing non-negative entries along the diagonal (the singular values).
- Define *S* by

$$S = \begin{cases} I, & \text{if } \det U \det V = 1 \\ \text{diag}(1, \ldots, 1, -1), & \\ \text{otherwise} & \end{cases}$$

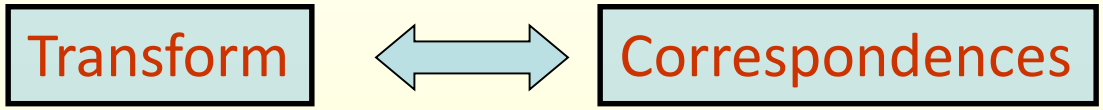
- Then $\boxed{R = USV^T}$

*SVD = singular value decomposition

O(*n*) algorithm!

How to Get Correspondences?

A chicken-and-egg problem: if we knew the optimal aligning transform, then we could get correspondences by proximity (possibly with the aid of some global adjustment, e.g., dynamic programming)



Guess one, estimate the other, and *iterate!*

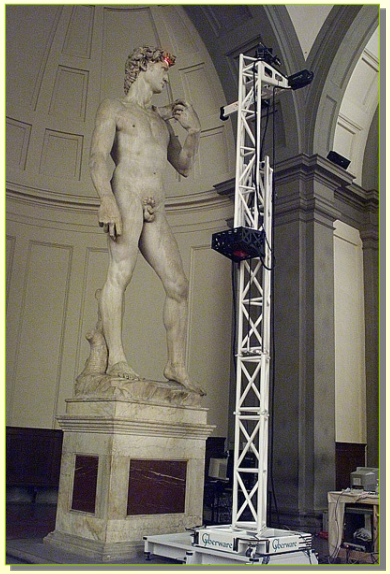
EM like

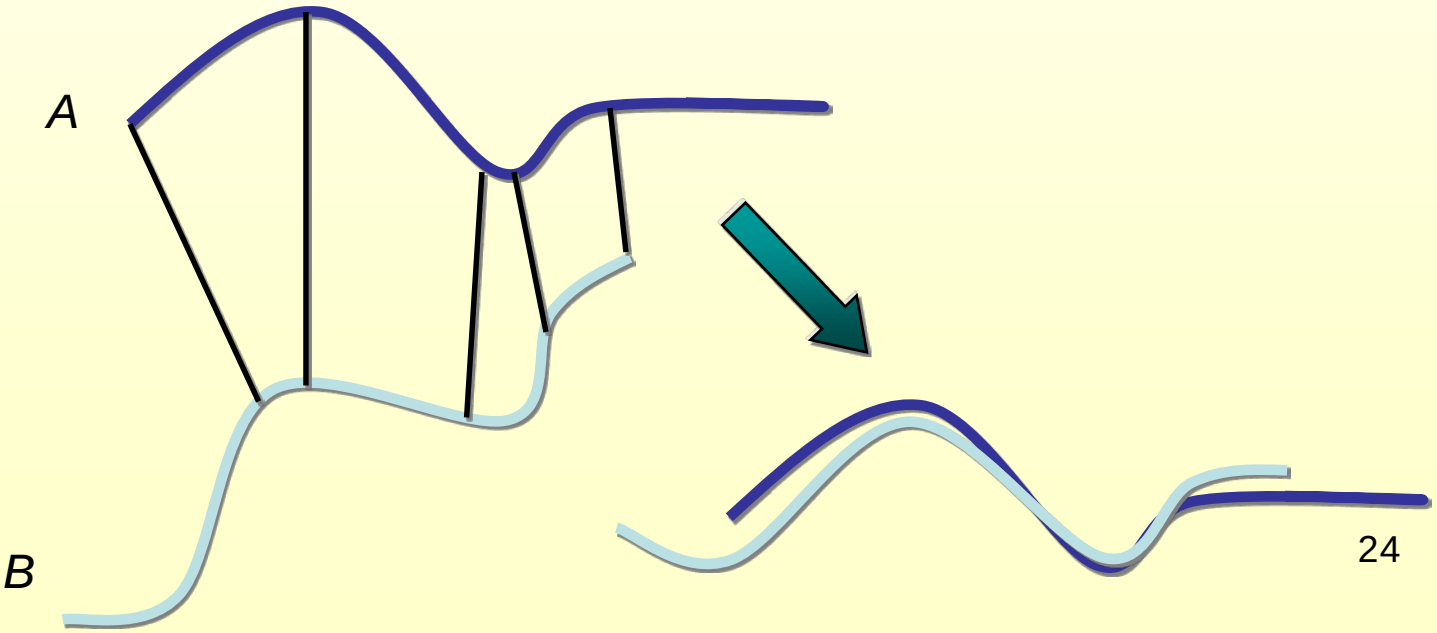
- Correspondences from proximity (Iterated Closest Pair)
- Correspondences from local shape descriptors (Shape Features)
- Transform from voting schemes (RANSAC, Geometric Hashing)
- Combinations

Local Methods: Iterated Closest Pair (ICP) [Besl, McKay 1992]

Aligning 3D Data

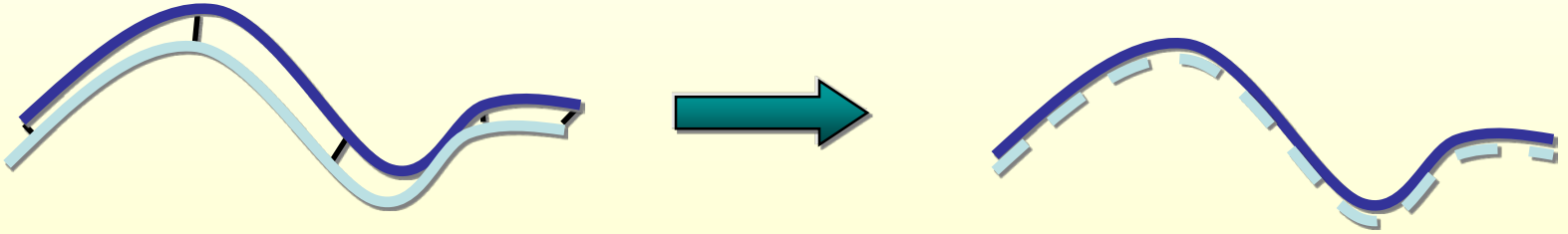
- How to find correspondences: User input? Feature detection? Local shape signatures?
- When *A* and *B* are partial scans of the same stationary object captured in a consistent setting, use the simplest alternative: assume closest points correspond

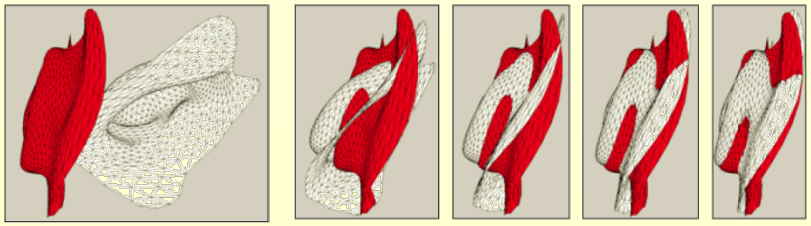


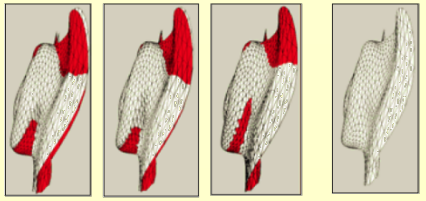


Aligning 3D Data

- Align the *A* points to their closest *B* neighbors, then repeat
- Converges, if starting positions are “close enough”



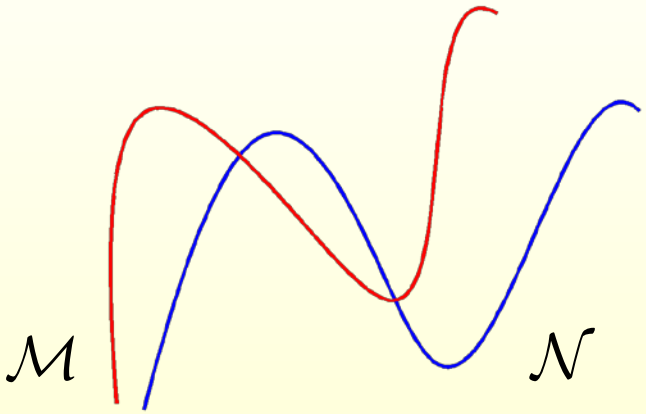




Successive iterations bring the objects closer together

Iterative Closest Point (ICP)

- Classical approach: iterate between finding correspondences and finding the transformation:



example in 2D

Given a pair of shapes, $\mathcal{M}$ and $\mathcal{N}$, iterate:

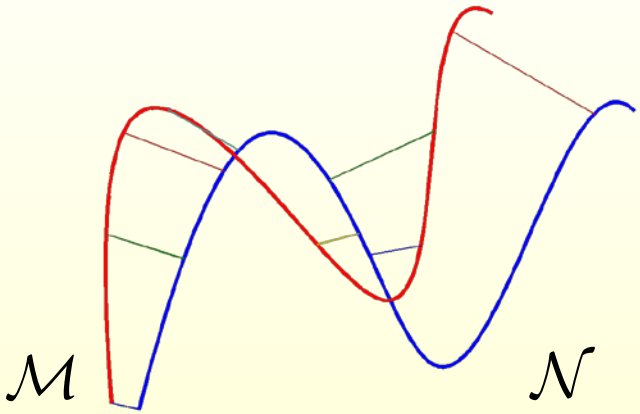
1. For each $x_i \in \mathcal{M}$ find **nearest** neighbor $y_i \in \mathcal{N}$
2. Find optimal transformation $\mathbf{R}, t$ minimizing:

$$\arg\min_{R,t} \sum_i \|Rx_i + t - y_i\|_2^2$$

Classic problem, solvable by SVD

Iterative Closest Point (ICP)

- Classical approach: iterate between finding correspondences and finding the transformation:



example in 2D

Given a pair of shapes, $\mathcal{M}$ and $\mathcal{N}$, iterate:

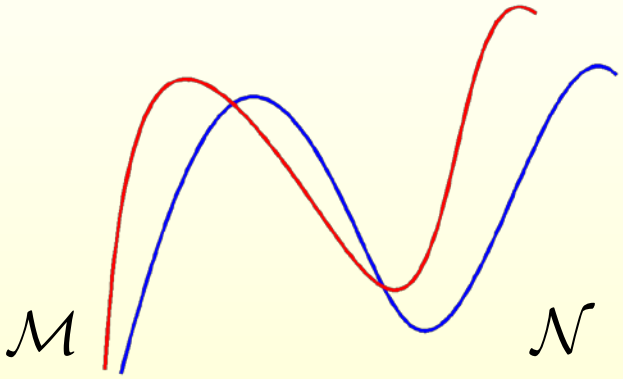
1. For each $x_i \in \mathcal{M}$ find **nearest** neighbor $y_i \in \mathcal{N}$
2. Find optimal transformation $\mathbf{R}, t$ minimizing:

$$\arg\min_{R,t} \sum_i \|Rx_i + t - y_i\|_2^2$$

Classic problem, solvable by SVD

Iterative Closest Point (ICP)

- Classical approach: iterate between finding correspondences and finding the transformation:



example in 2D

Given a pair of shapes, $\mathcal{M}$ and $\mathcal{N}$, iterate:

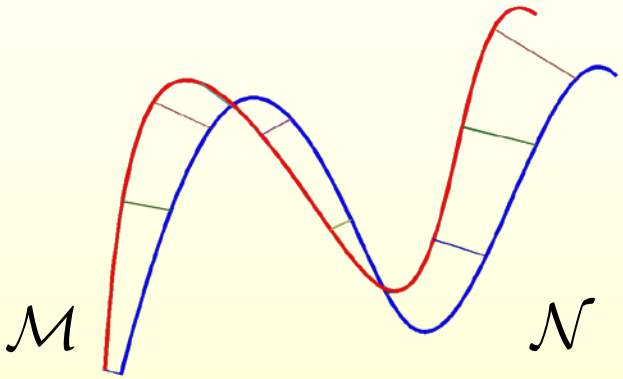
1. For each $x_i \in \mathcal{M}$ find **nearest** neighbor $y_i \in \mathcal{N}$
2. Find optimal transformation $\mathbf{R}, t$ minimizing:

$$\arg\min_{R,t} \sum_i \|Rx_i + t - y_i\|_2^2$$

Classic problem, solvable by SVD

Iterative Closest Point (ICP)

- Classical approach: iterate between finding correspondences and finding the transformation:



example in 2D

Given a pair of shapes, $\mathcal{M}$ and $\mathcal{N}$, iterate:

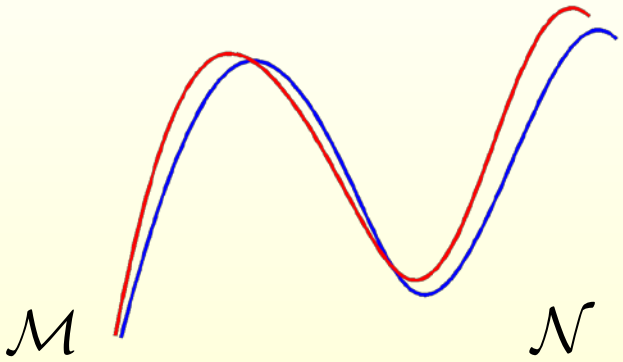
1. For each $x_i \in \mathcal{M}$ find **nearest** neighbor $y_i \in \mathcal{N}$
2. Find optimal transformation $\mathbf{R}, t$ minimizing:

$$\arg\min_{R,t} \sum_i \|Rx_i + t - y_i\|_2^2$$

Classic problem, solvable by SVD

Iterative Closest Point (ICP)

- Classical approach: iterate between finding correspondences and finding the transformation:



example in 2D

Given a pair of shapes, $\mathcal{M}$ and $\mathcal{N}$, iterate:

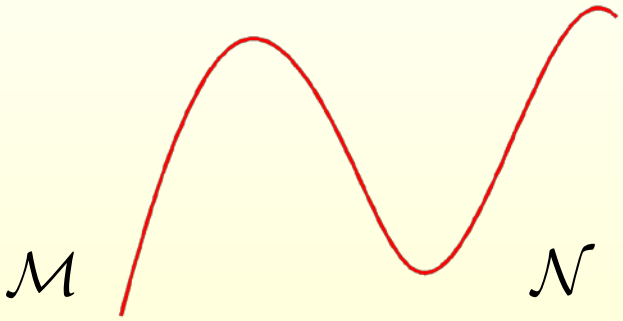
1. For each $x_i \in \mathcal{M}$ find **nearest** neighbor $y_i \in \mathcal{N}$
2. Find optimal transformation $\mathbf{R}, t$ minimizing:

$$\arg\min_{R,t} \sum_i \|Rx_i + t - y_i\|_2^2$$

Classic problem, solvable by SVD

Iterative Closest Point (ICP)

- Classical approach: iterate between finding correspondences and finding the transformation:



example in 2D

Given a pair of shapes, $\mathcal{M}$ and $\mathcal{N}$, iterate:

1. For each $x_i \in \mathcal{M}$ find **nearest** neighbor $y_i \in \mathcal{N}$
2. Find optimal transformation $\mathbf{R}, t$ minimizing:

$$\arg\min_{R,t} \sum_i \|Rx_i + t - y_i\|_2^2$$

Classic problem, solvable by SVD

Convergence Theorem

- The ICP algorithm always converges monotonically to a local minimum, with respect to the MSE distance objective function
  - Correspondence step improves error bound – because of nearest neighbor computation
  - Rotation/Translation step improves error bound – because of transform optimization

Time Analysis

Each iteration includes three main steps

A. Finding the closest points:

$O(N_M)$ per point

$O(N_M * N_S)$ total

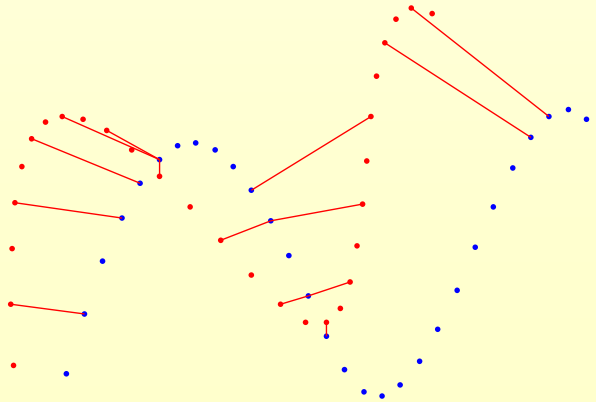
B. Calculating the optimal alignment: $O(N_S)$

C. Updating the scene: $O(N_S)$

Fast nearest-neighbor data structures can be very helpful here, e.g., a *k-d* tree

Variations of ICP

1. Selecting source points (from one or both scans): sampling
2. Matching to points in the other mesh
3. Weighting the correspondences
4. Rejecting certain (outlier) point pairs
5. Assigning an error metric to the current transform
6. Minimizing the error metric w.r.t. the transformation

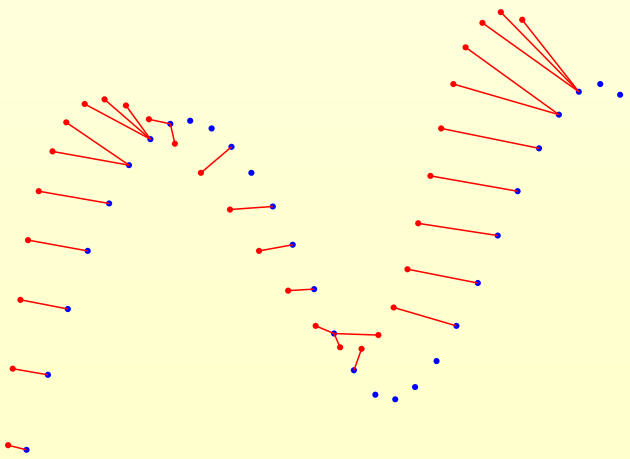


Iterative Closest Point

Given a pair of shapes, X and Y, iterate:

1. For each $x_i \in X$ find **nearest** neighbor $y_i \in Y$.
2. Find deformation $\mathbf{R}, t$ minimizing:

$$\sum_{i=1}^{N} \|\mathbf{R}x_i + t - y_i\|_2^2$$



Ideally, most correspondences are 1-1

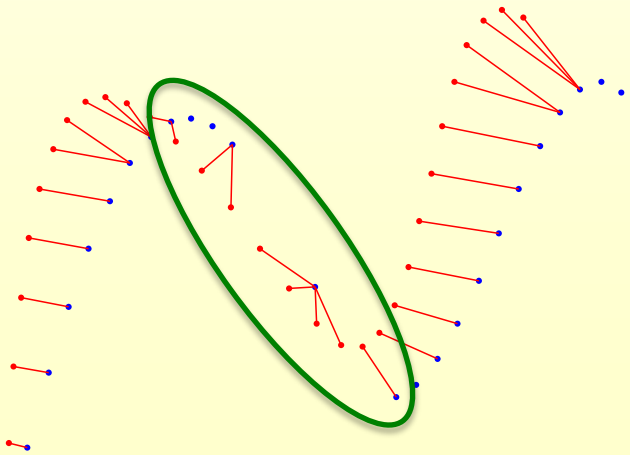
Iterative Closest Point

Given a pair of shapes, X and Y, iterate:

1. For each $x_i \in X$ find **nearest** neighbor $y_i \in Y$.
2. Find deformation $\mathbf{R}, t$ minimizing:

$$\sum_{i=1}^{N} \|\mathbf{R}x_i + t - y_i\|_2^2$$

Problem:
uneven sampling



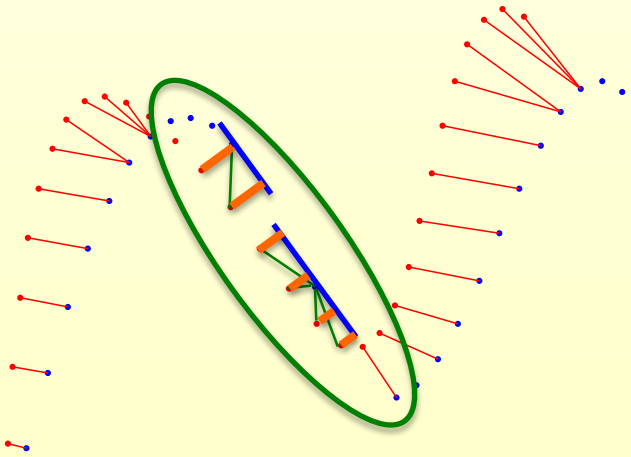
Iterative Closest Point

Given a pair of shapes, X and Y, iterate:

1. For each $x_i \in X$ find **nearest** neighbor $y_i \in Y$.
2. Find deformation $\mathbf{R}, t$ minimizing:

$$\sum_{i=1}^{N} d(\mathbf{R}x_i + t, P(y_i))^2 = \sum_{i=1}^{N} \left((\mathbf{R}x_i + t - y_i)^T \mathbf{n}_{y_i}\right)$$

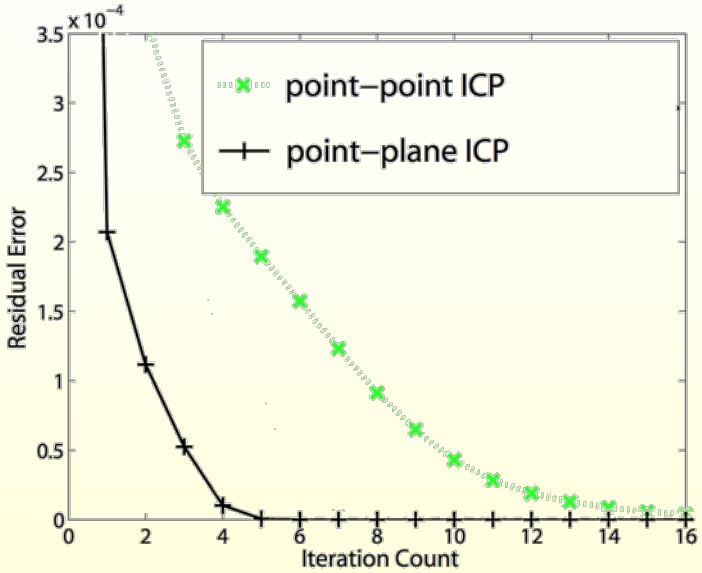
Solution:
Minimize distance to the tangent plane

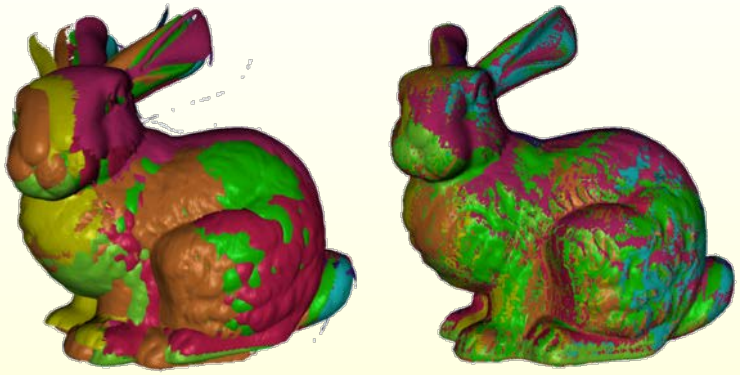


No longer a closed-form solution

Chen, Medioni, '91

Iterative Closest Point



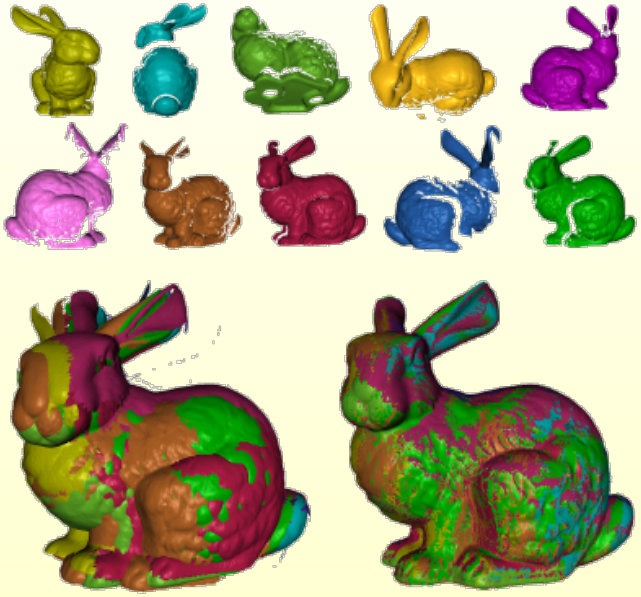


Aligning the bunny to itself:
Point-to-plane always wins in the end-game.

Global Matching Methods

Global Matching

Given shapes in *arbitrary* positions, find their alignment:



Robust Global Registration
Gelfand et al. SGP 2005

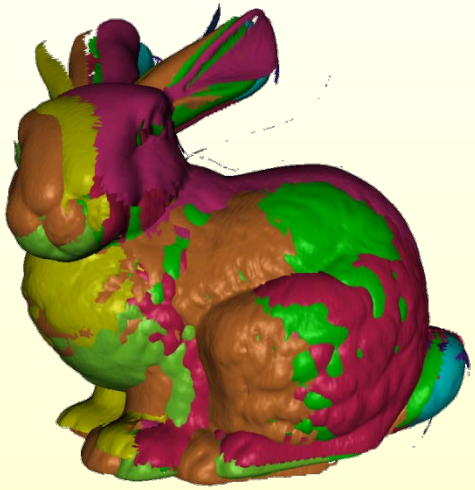
Can be approximate, since will refine later using e.g. ICP

Partial Alignment

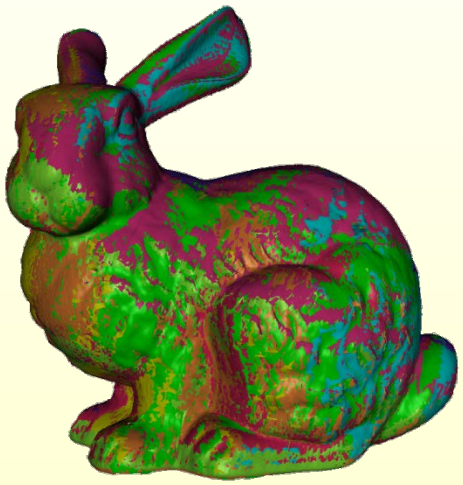
After 6 iterations

Multiple Alignment Results

Input: 10 scans



Approximate alignment using B&B



Refined by ICP

Bundle adjustment

Global Matching – Approaches

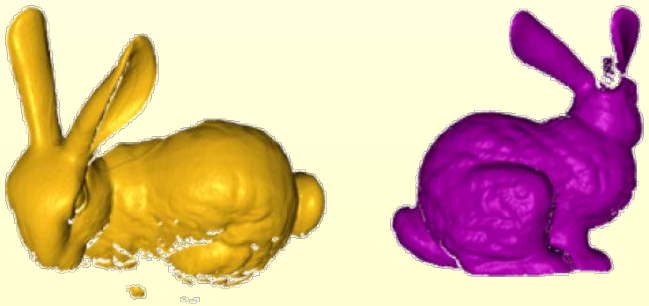
Several classes of approaches:

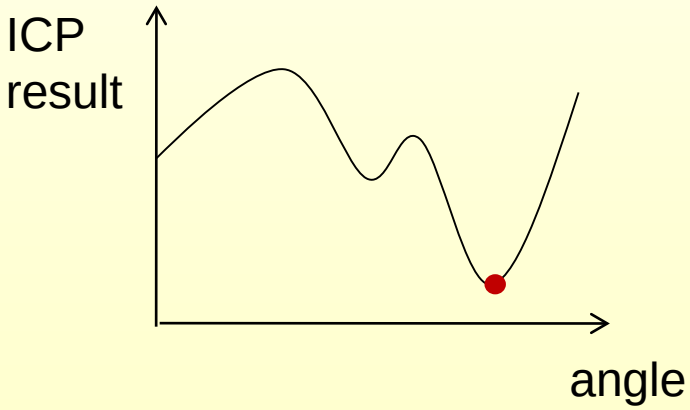
1. Exhaustive Search
2. Normalization
3. Random Sampling
4. Invariance

Exhaustive Search:

Compare (ideally) all alignments

- Sample the space of possible initial alignments.
- Correspondence is determined by the alignment at which models are closest.





Very common in biology: e.g., protein docking

Exhaustive Search:

Compare at all alignments

- Sample the space of possible initial alignments
- Correspondence is determined by the alignment at which models are closest

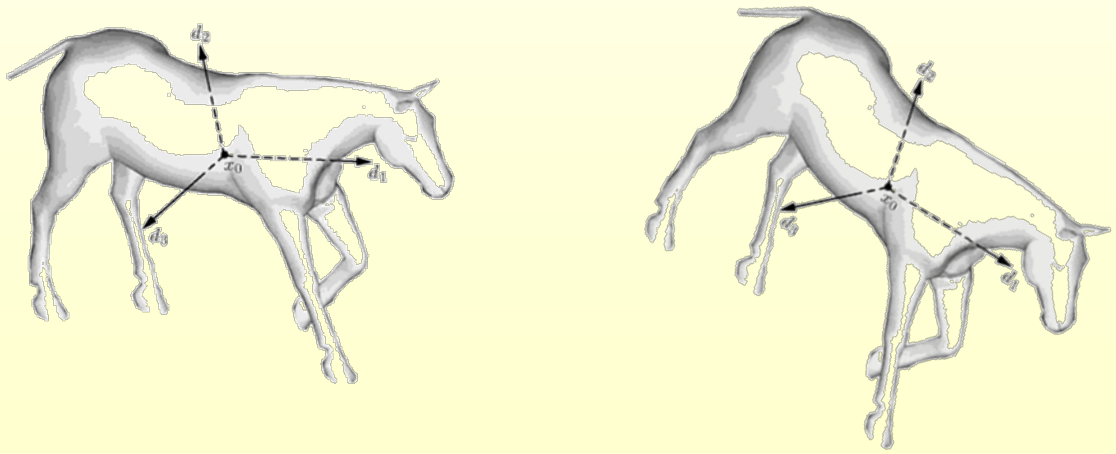
- Provides optimal result
- Can be unnecessarily slow
- Does not generalize well to non-rigid deformations

Normalization – Canonical Poses

There are only a handful of initial configurations

that are important.

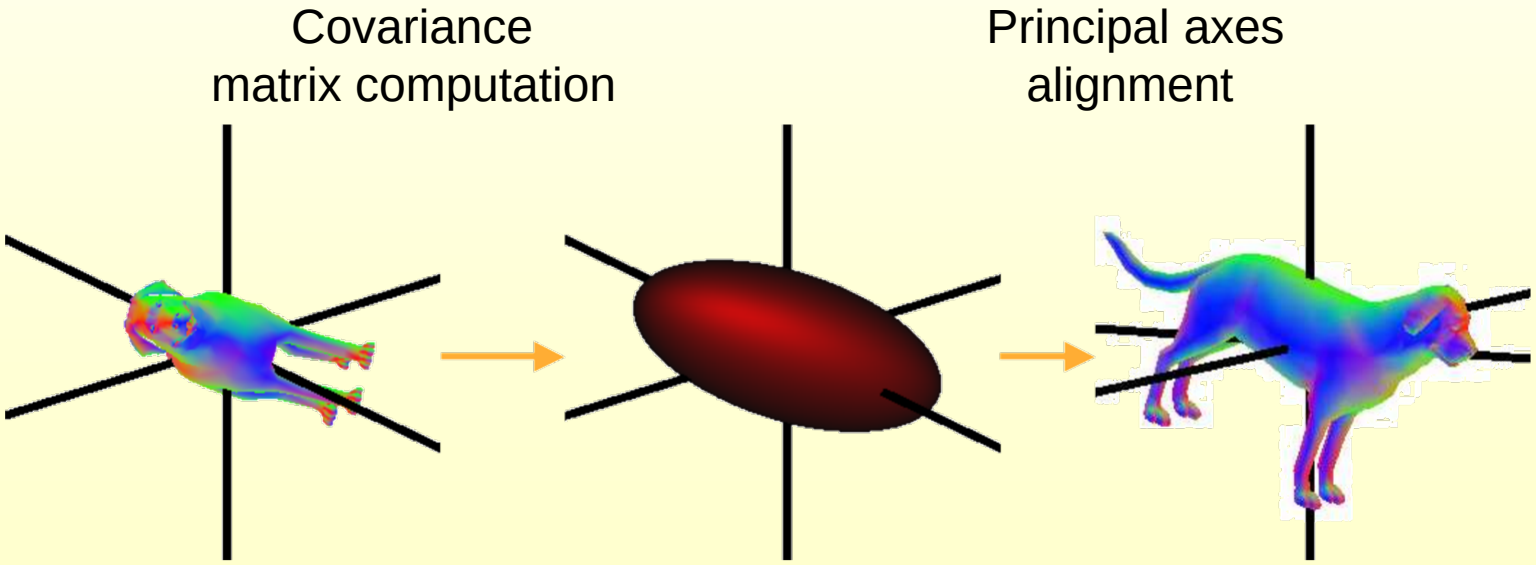
Can center all shapes at the origin and use PCA to find the principal directions of the shape.



In addition sometimes try all permutations of x-y-z.

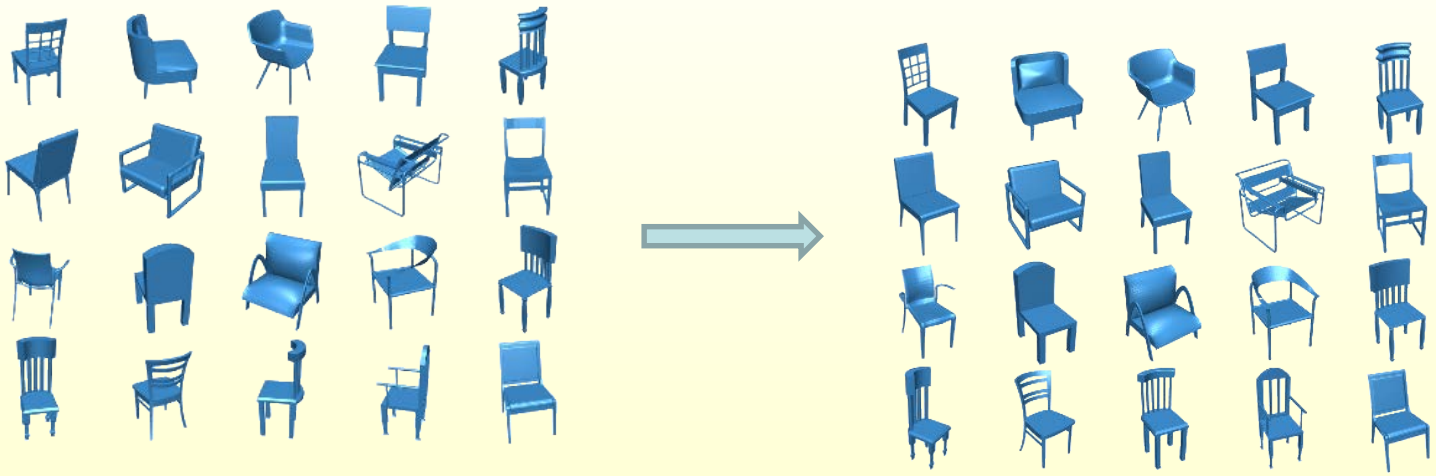
PCA-Based Alignment

- Use PCA to place models into canonical coordinate frames
- Then align those frames



Normalization – Canonical Poses

There are only a handful of initial configurations that are important.

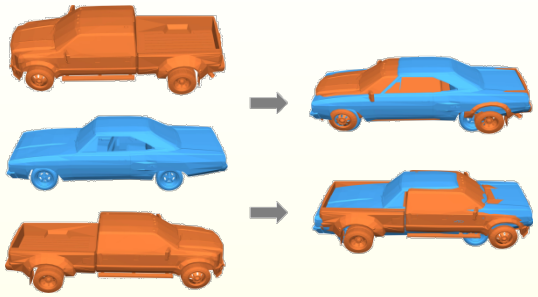


Works well if we have complete shapes and no noise.

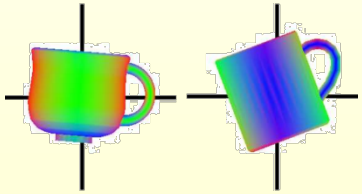
Fails for partial scans, outliers, high noise, etc.

Problems with PCA

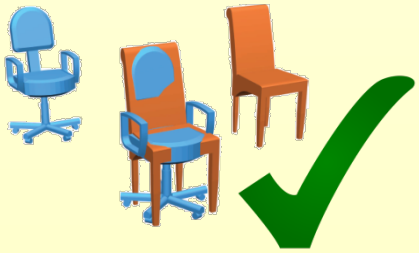
- Principal axes are not consistently oriented



- Axes are unstable when principal values are similar



- Partial similarity

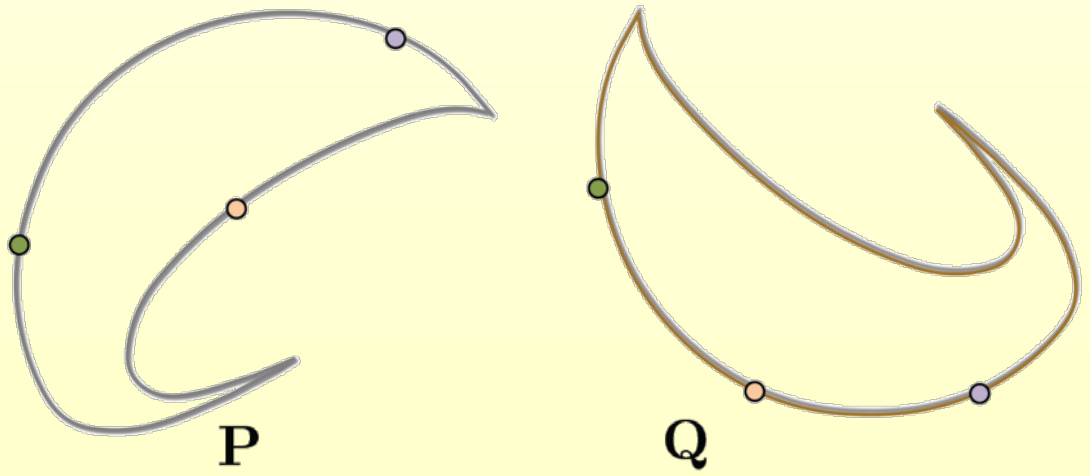


Random Sampling (RANSAC)

ICP only needs 3 point pairs!

Robust and Simple approach. Iterate between:

1. Pick a random pair of 3 points on model & scan
2. Estimate alignment, and check for error.



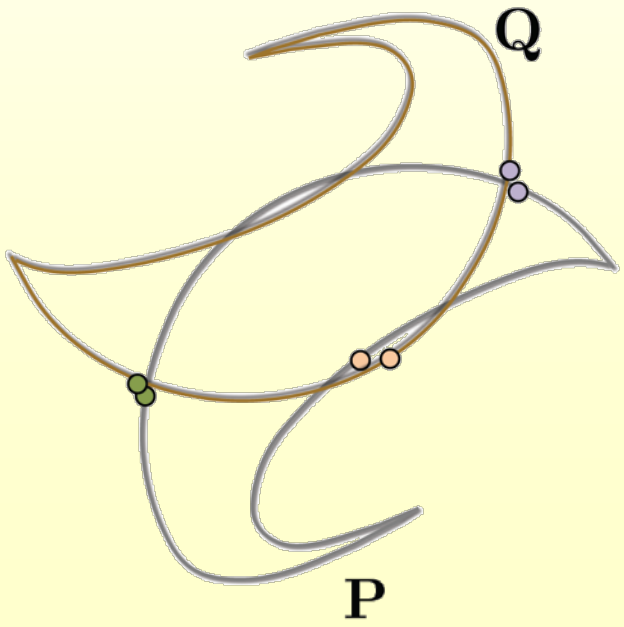
Guess and verify

Random Sampling (RANSAC)

ICP only needs 3 point pairs!

Robust and simple approach. Iterate between:

1. Pick a random pair of 3 points on model & scan
2. Estimate alignment, and check for error.



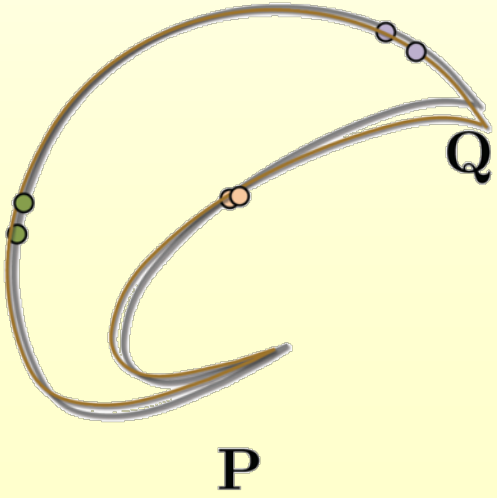
Guess and verify

Random Sampling (RANSAC)

ICP only needs 3 point pairs!

Robust and simple approach. Iterate between:

1. Pick a random pair of 3 points on model & scan
2. Estimate alignment, and check for error.



Can also refine the final result. Picks don't have to be exact.

Cab be expensive!

Guess and verify

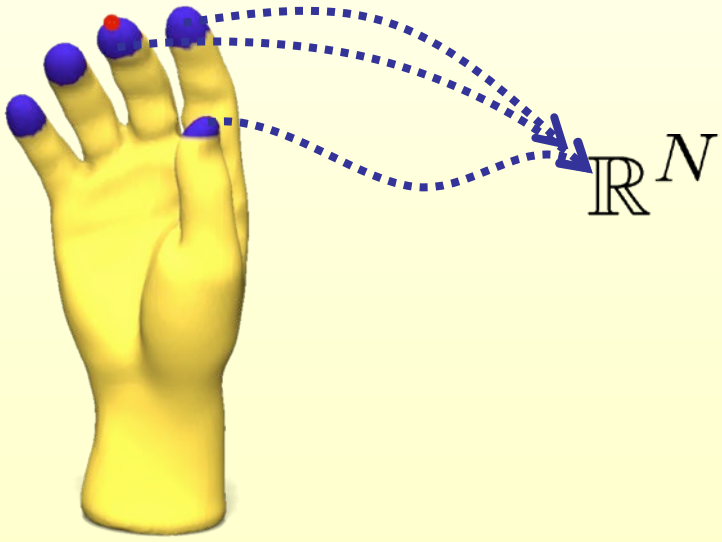
Global Matching – Invariant Features

Try to characterize the shape using properties that are invariant under the desired set of transformations.

Conflicting interests – invariance vs. informativeness.

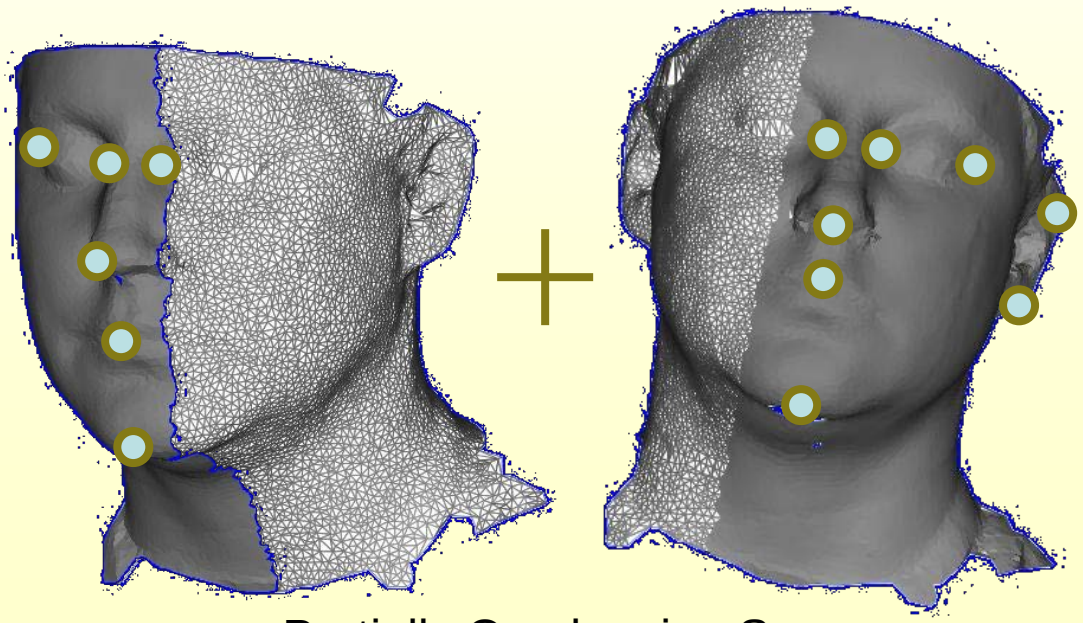
The most common pipeline:

1. identify salient feature points
2. compute informative and commensurable descriptors.



Matching Using Feature Points

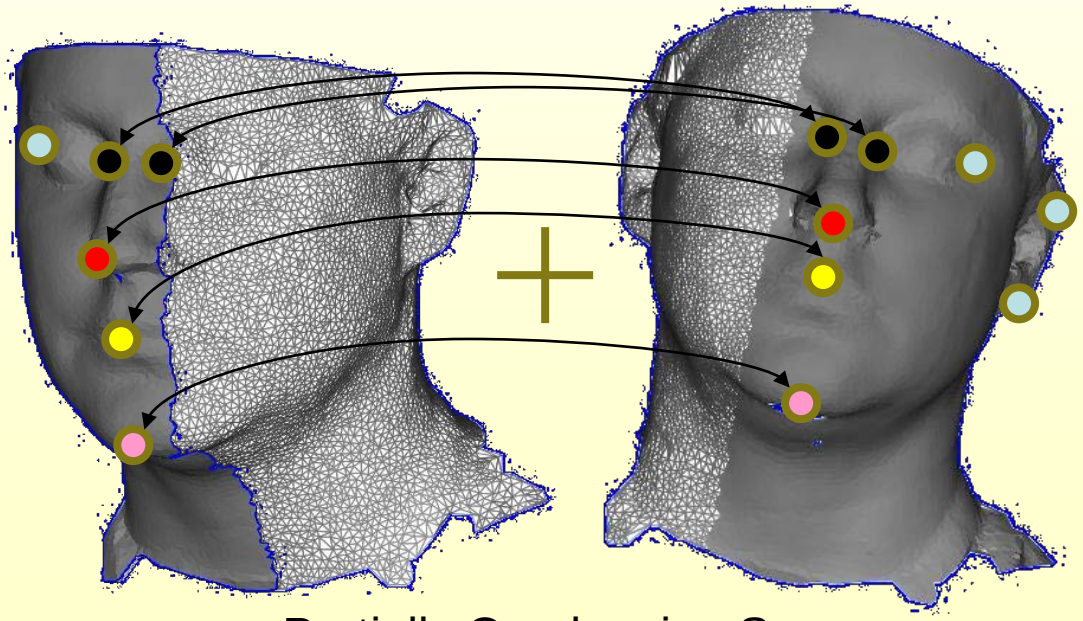
1. Find feature points on the two scans (we'll come back to that issue)



Partially Overlapping Scans

Approach

1. (Find feature points on the two scans)
2. Establish correspondences

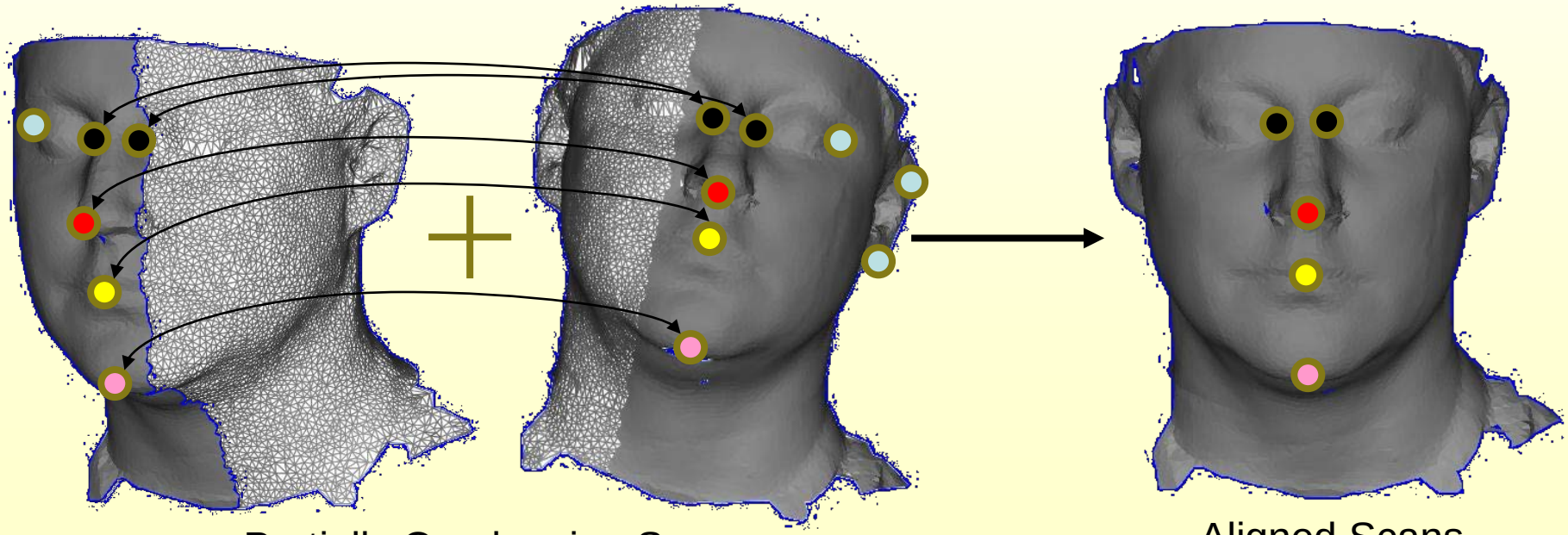


Partially Overlapping Scans

Approach

1. (Find feature points on the two scans)
2. Establish correspondences
3. Compute the aligning transformation

Preserve features
Various regularizers



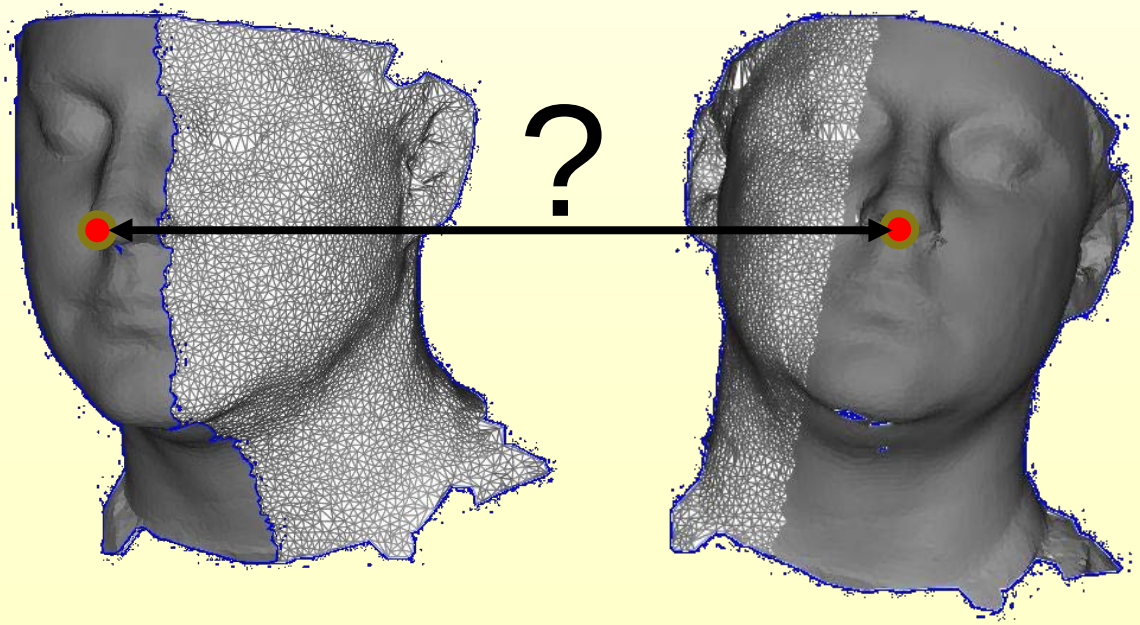
Partially Overlapping Scans

Aligned Scans

Correspondence

Goal:

Identify when two points on different scans represent the same feature

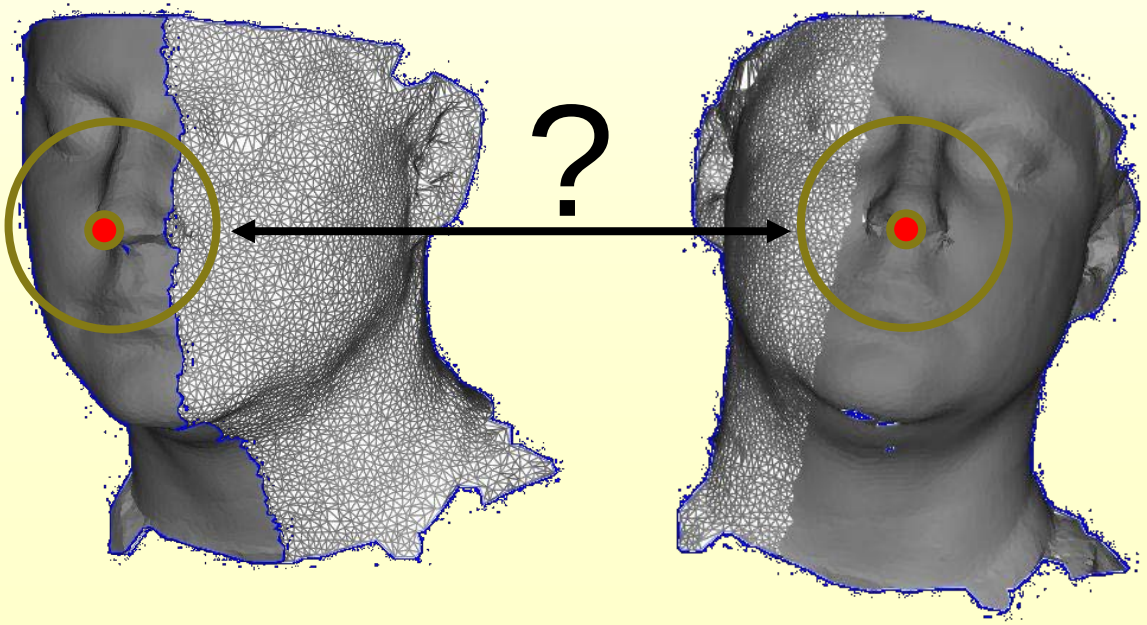


Correspondence

Goal:

Identify when two points on different scans represent the same feature:

Are the surrounding regions similar?

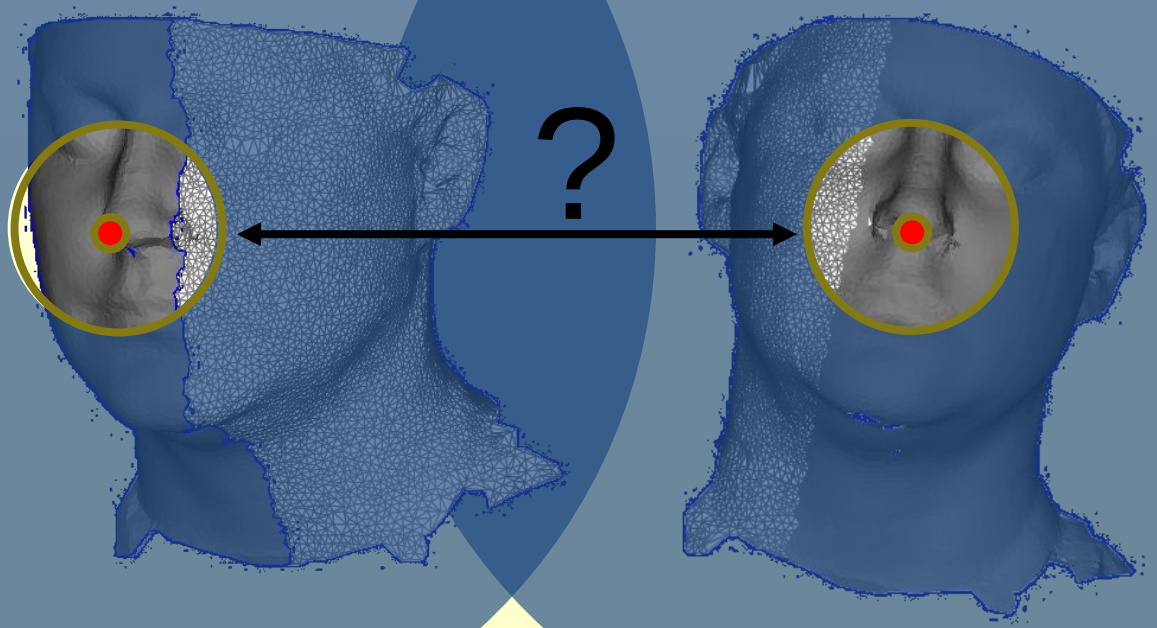


Correspondence

Goal:

Identify when two points on different scans represent the same feature:

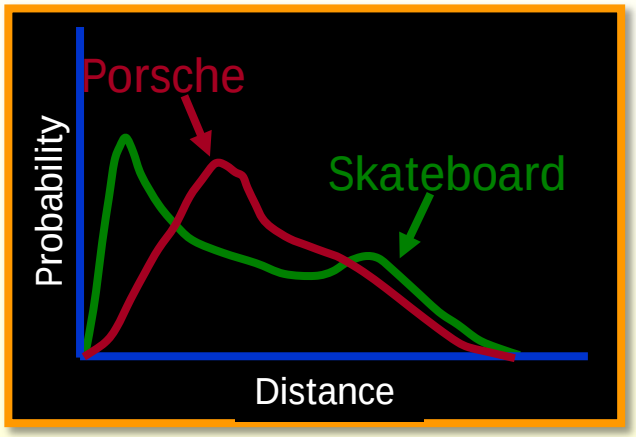
Are the surrounding regions similar?

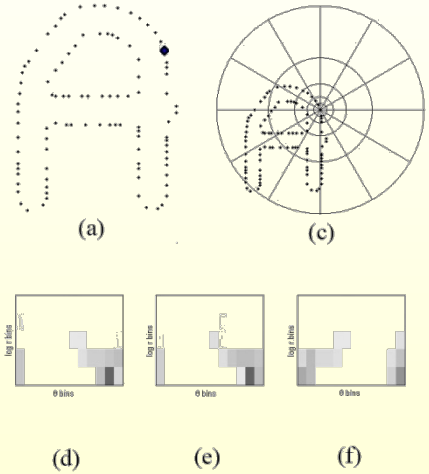


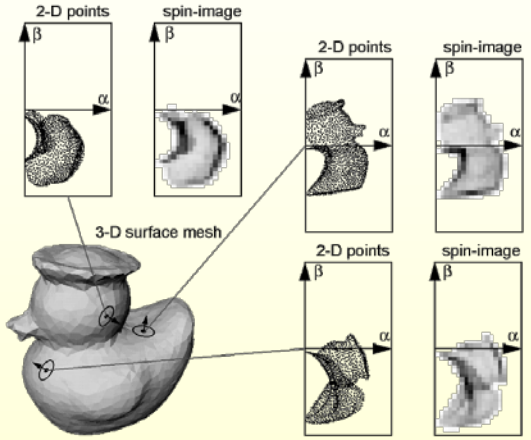
Shape Descriptors

A Mapping Tool: Descriptors of Shapes, Shape Parts, Shape Points

- For shapes, there are many descriptors with various types of invariances







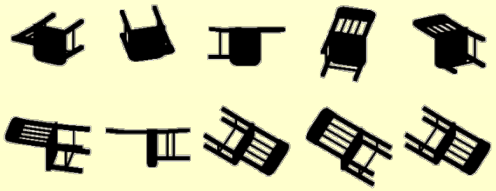
D2 Distribution:
[Osada et al. '01]

Shape Contexts:
[Belongie et al. '00, Frome et al. '04]

Spin Images:
[Johnson, Hebert '99]

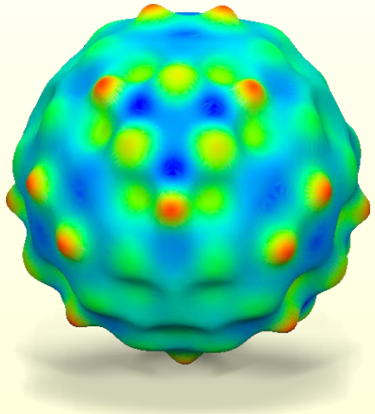
- Local or global?
- Intrinsic or extrinsic?

Lightfield:
[Chen et al. '03]



Classical Curvature

- Differential features can be noisy on meshes

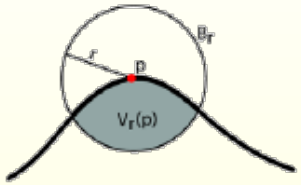


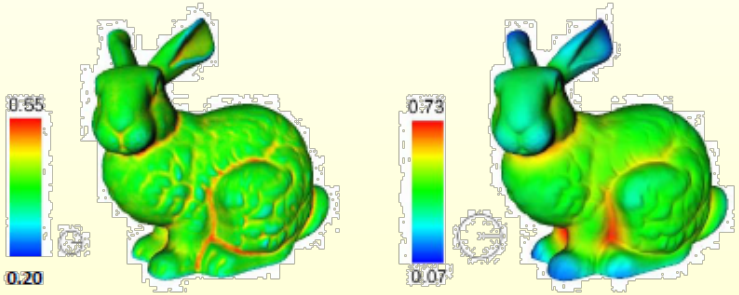
Curvature

Integral Volume Descriptor

Integral invariant signatures, Manay et al. ECCV 2004
Integral Invariants for Robust Geometry Processing, Pottmann et al. 2007-2009

$$V_r(p) = \int_{B_r(p) \cap S} dx$$





Relation to mean curvature

$$V_r(\mathbf{p}) = \frac{2\pi}{3} r^3 - \frac{\pi H}{4} r^4 + O(r^5)$$

Robust Global Registration, Gelfand et al. 2005

Spin Images

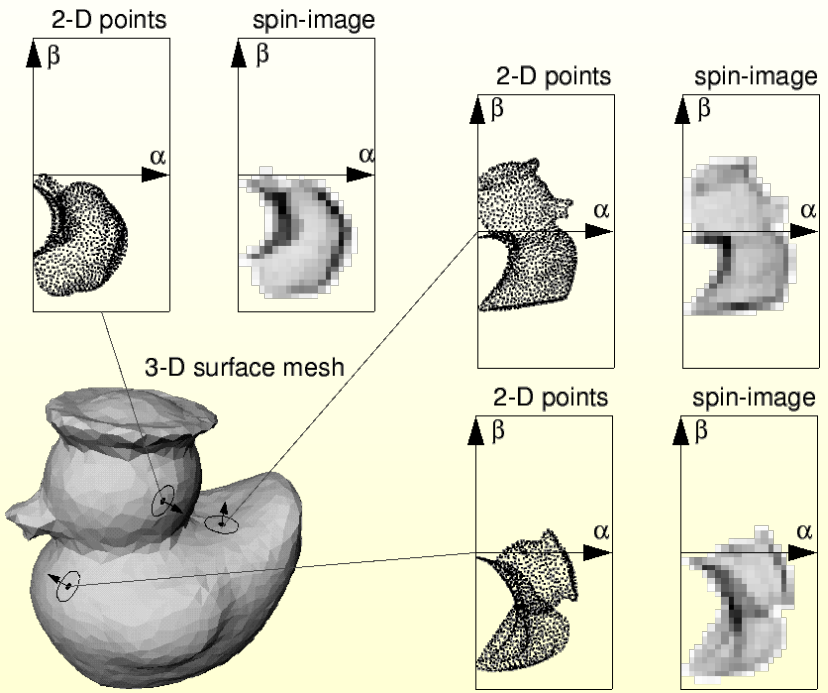
Creates an image associated with a neighborhood of a point.

Compare points by comparing their *spin images* (2D).

Given a point and a normal, every other point is indexed by two parameters:

β distance to tangent plane

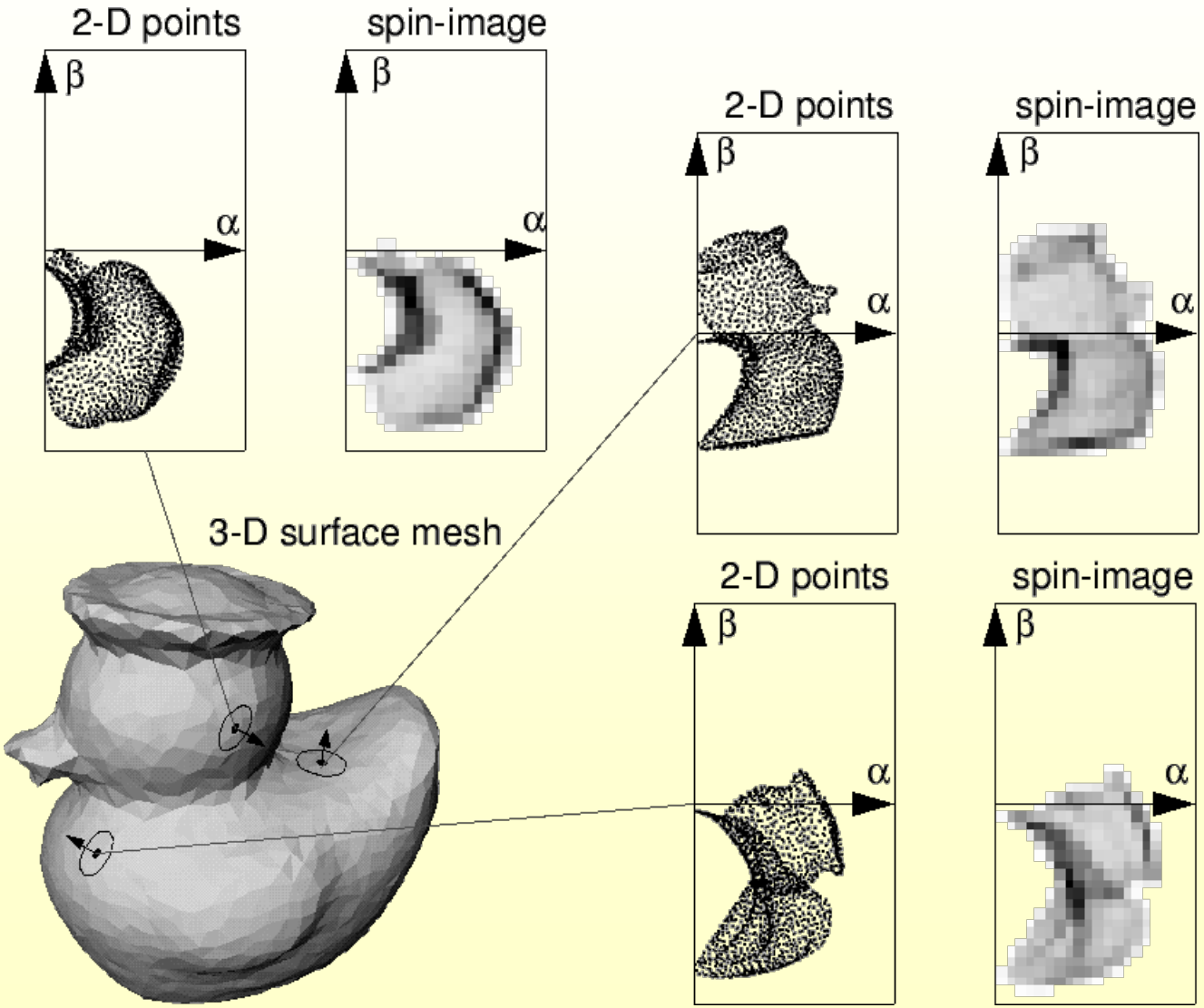
α distance to normal line



Using Spin Images for Efficient Object Recognition in Cluttered 3D Scenes
Johnson et al, PAMI 99

α - radial dist.
β - elevation

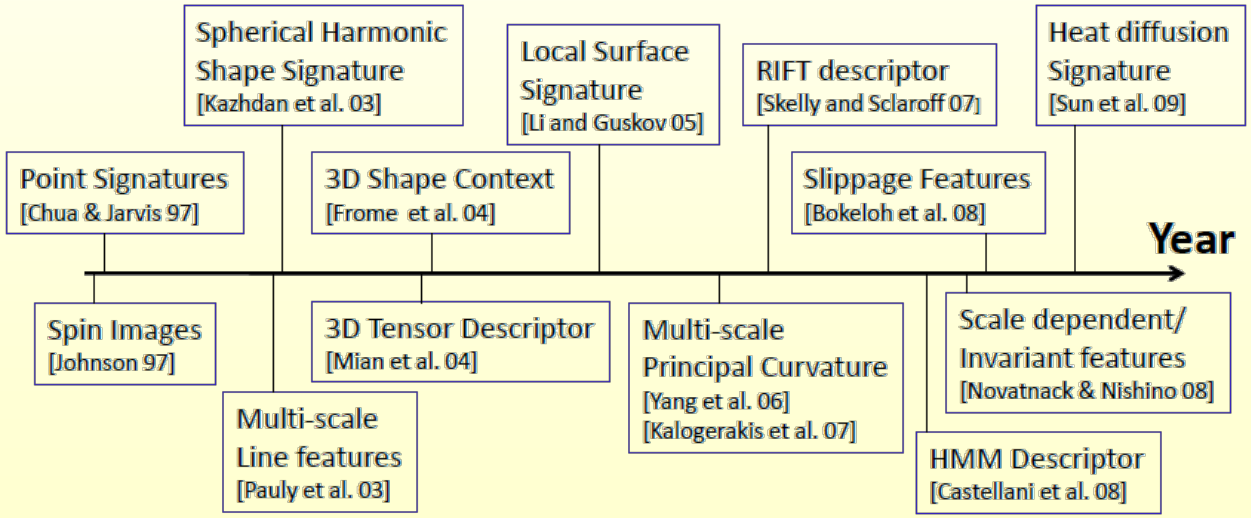
Spin Images



Main Question

How to compare regions on the shape in an invariant manner?

A large variety of *descriptors* have been suggested.



To give an example, we describe two.

table by Will Chang

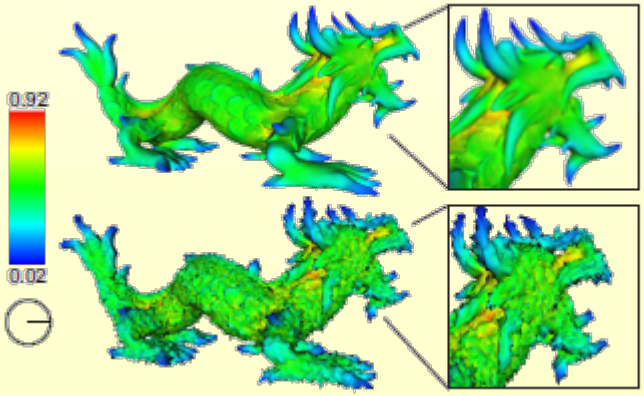
Feature Based Methods

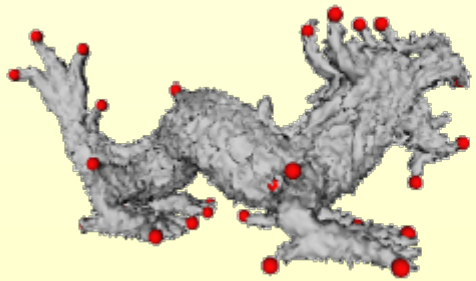
Once we have a feature descriptor, we can find the most *unusual* one: feature detection.

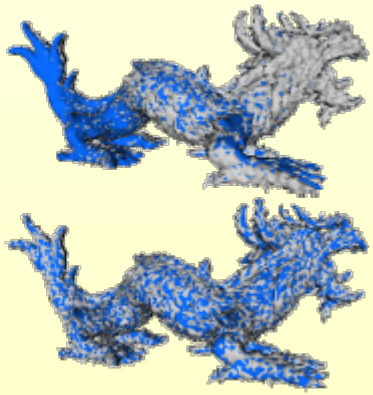
Establish correspondences by first finding *reliable* ones.

Propagate the matches everywhere.

To backtrack use branch-and bound.







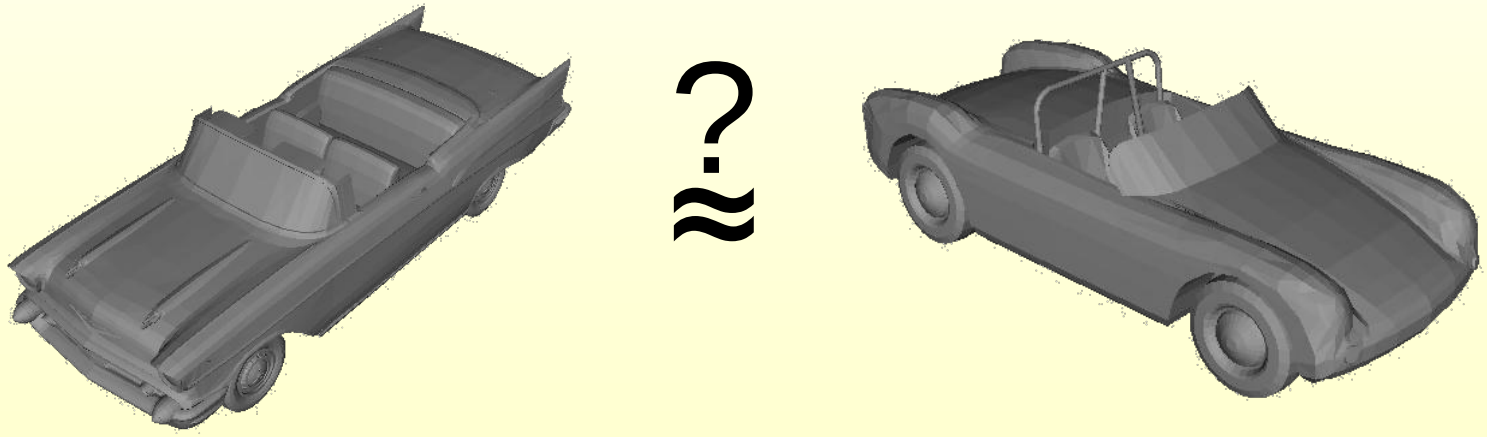
Robust Global Registration, Gelfand et al. 2005

Global Shape Similarity

Global Similarity

More Generally:

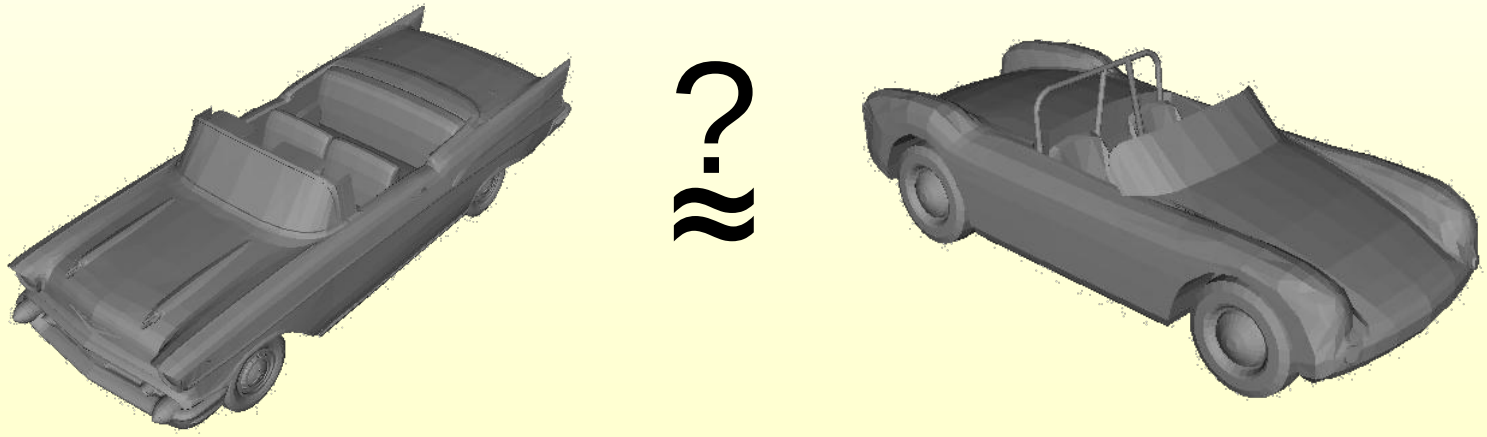
Given two 3D models, determine if they represent the same/similar shapes



Global Similarity

More Generally:

Given two models, determine if they represent the same/similar shapes.



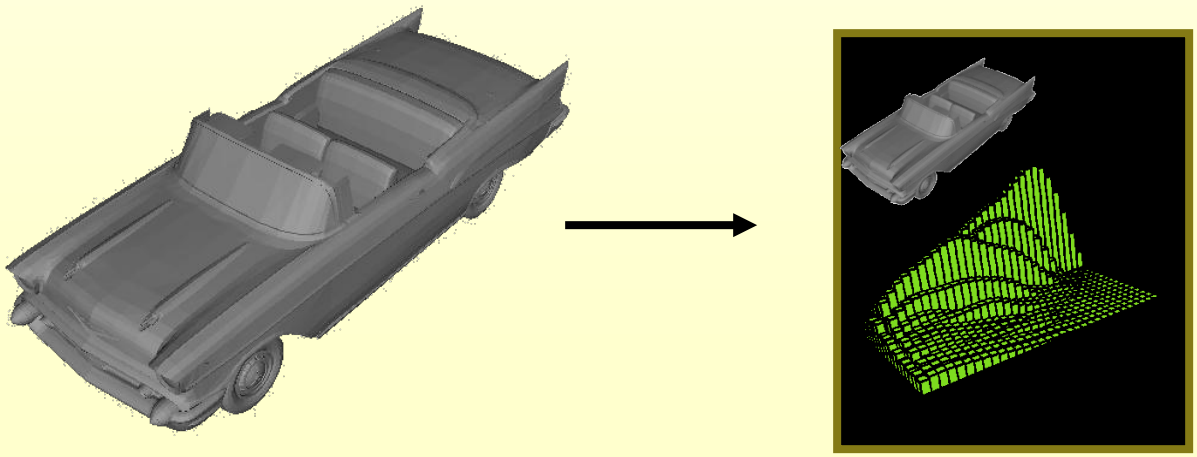
Models can have different:
representations, tessellations, topologies, etc.

Global Similarity

Approach:

1. Represent each model by a shape descriptor:
   - A structured abstraction of a 3D model
   - That captures salient shape information

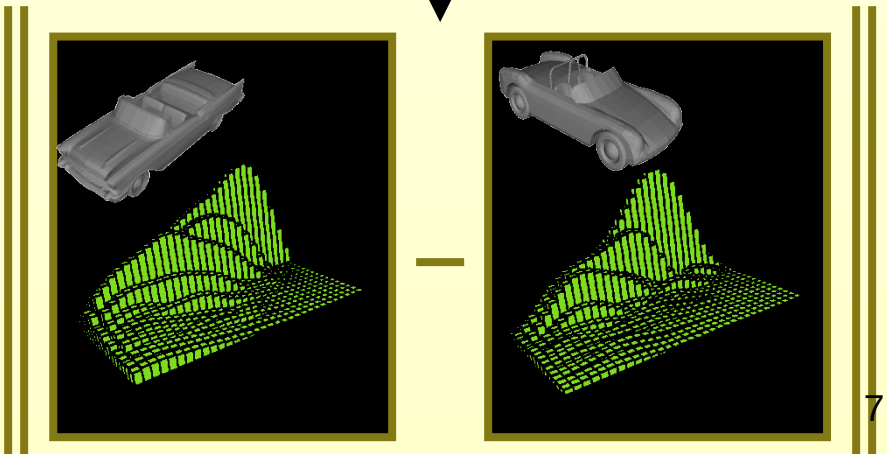
Typically a high-dimensional vector



Global Similarity

Approach:

1. Represent each model by a shape descriptor
2. Compare shapes by comparing their shape descriptors
3. Is the descriptor designed, or learned?



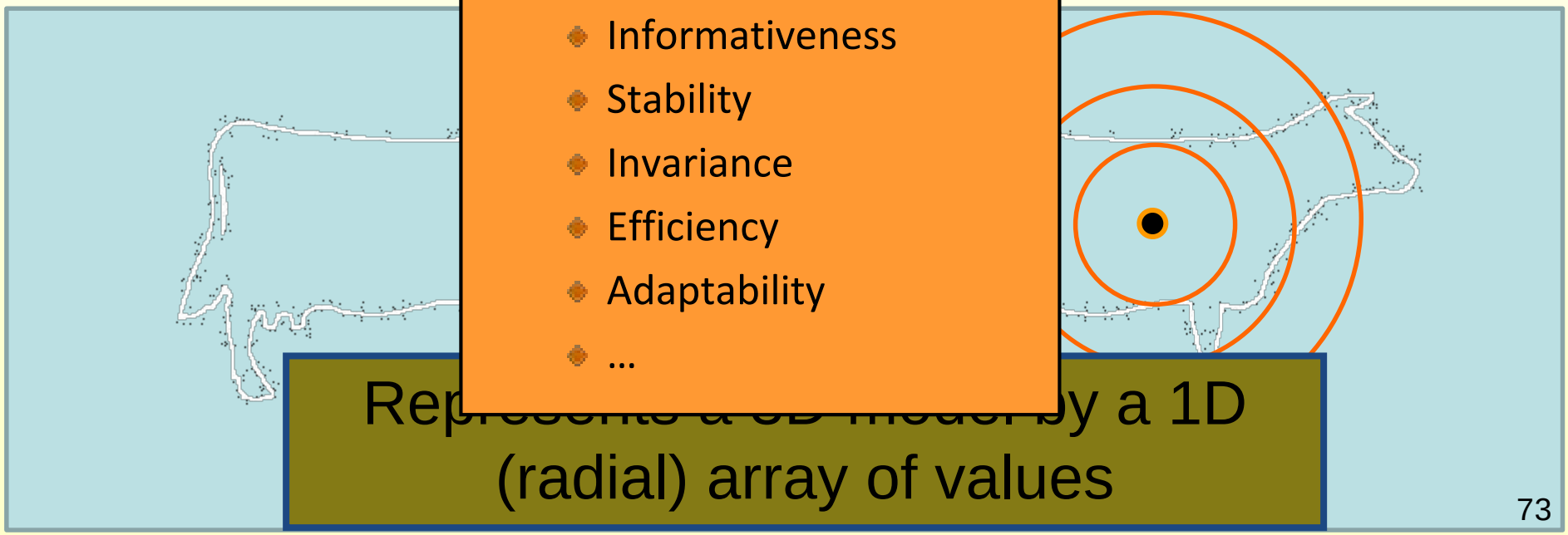
Shape Descriptors: Examples

[Ankerst *et al*. 1999]

Shape Histograms

Shape descriptor stores a histogram of how much surface area resides within different concentric shells in space

- Feature desiderata
  - Informativeness
  - Stability
  - Invariance
  - Efficiency
  - Adaptability
  - …

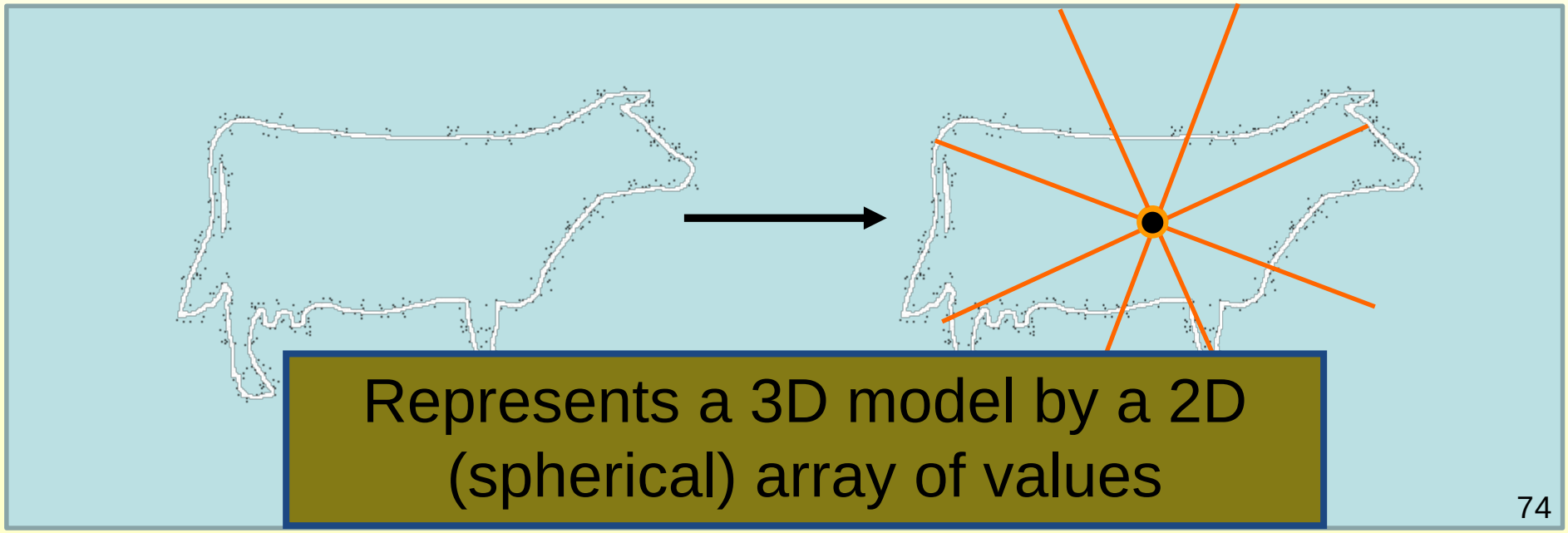


Shape Descriptors: Examples

[Ankerst *et al*. 1999]

Shape Histograms

Shape descriptor stores a histogram of how much surface area resides within different sectors in space

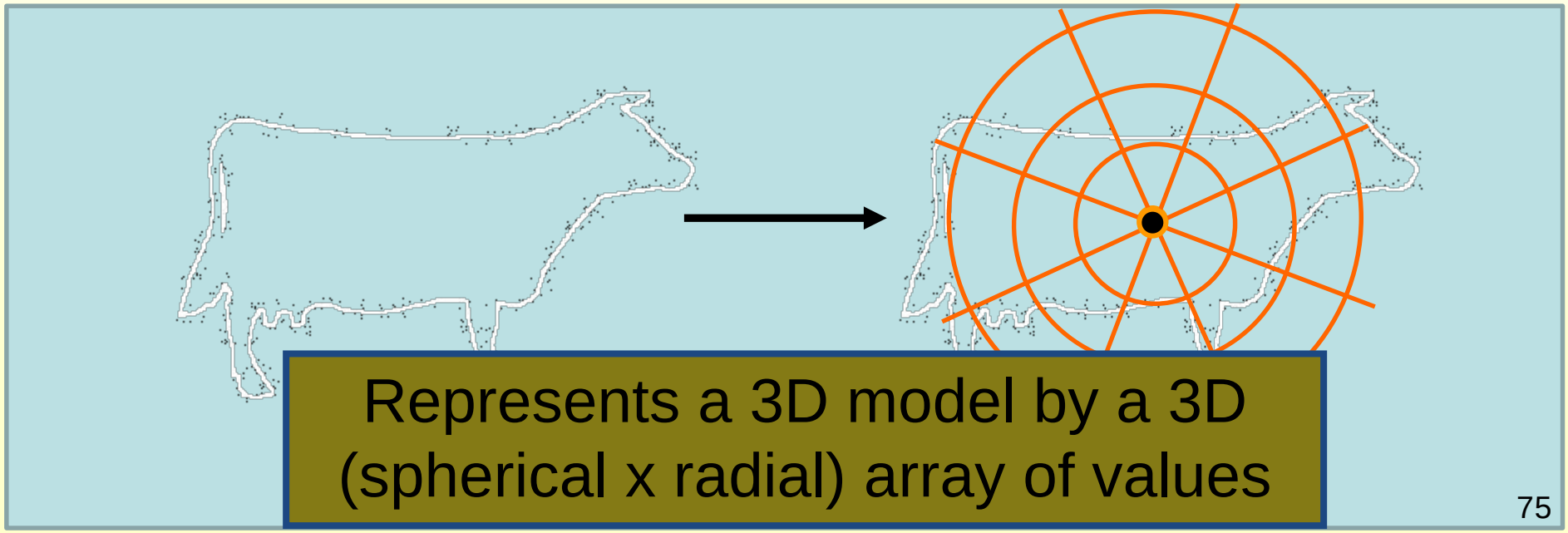


Shape Descriptors: Examples

[Ankerst *et al*. 1999]

Shape Histograms

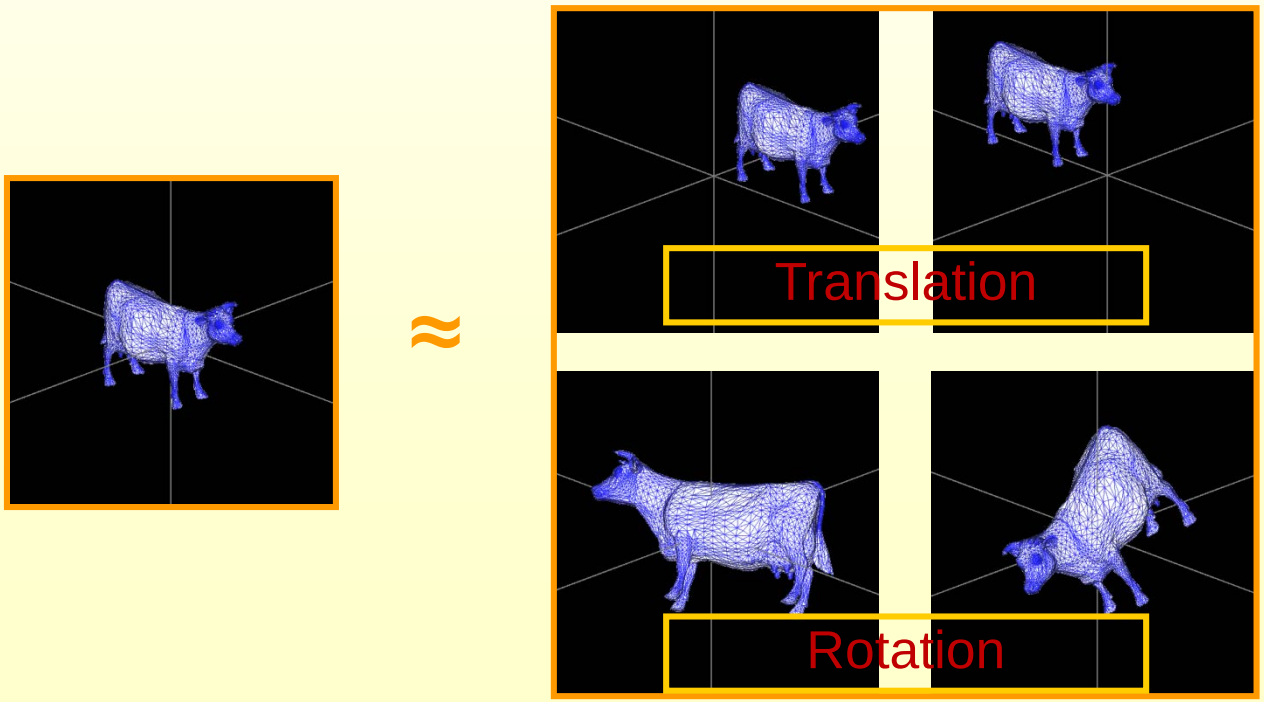
Shape descriptor stores a histogram of how much surface area resides within different shells and sectors in space



Represents a 3D model by a 3D (spherical x radial) array of values

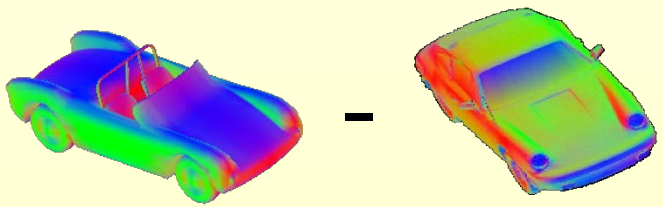
Shape Descriptors: Challenge

The descriptor must not change when a rigid body transformation (e.g., translation, rotation) is applied to the model



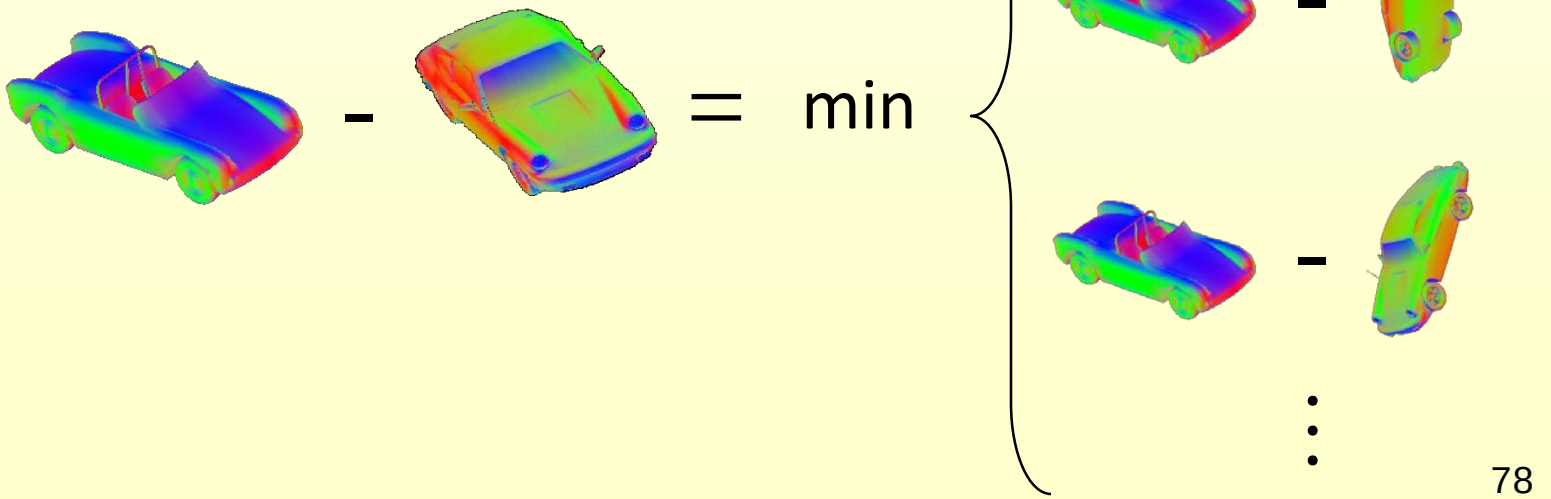
Shape Descriptors: Challenge

In order to compare two 3D models, we must either make descriptors invariant to rigid motions, or we need to compare the shapes at their optimal alignment



Shape Descriptors: Challenge

In order to compare two models,
we need to compare them
at their optimal alignment

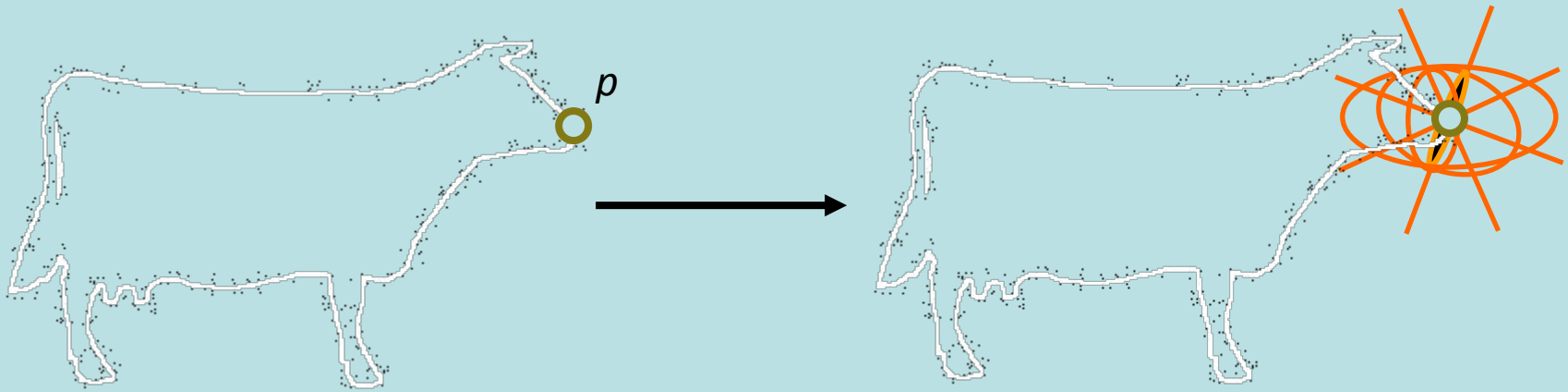


Local Features and Local Shape Similarity

From Global to Local

To characterize the surface about a point *p*, take a global descriptor and:

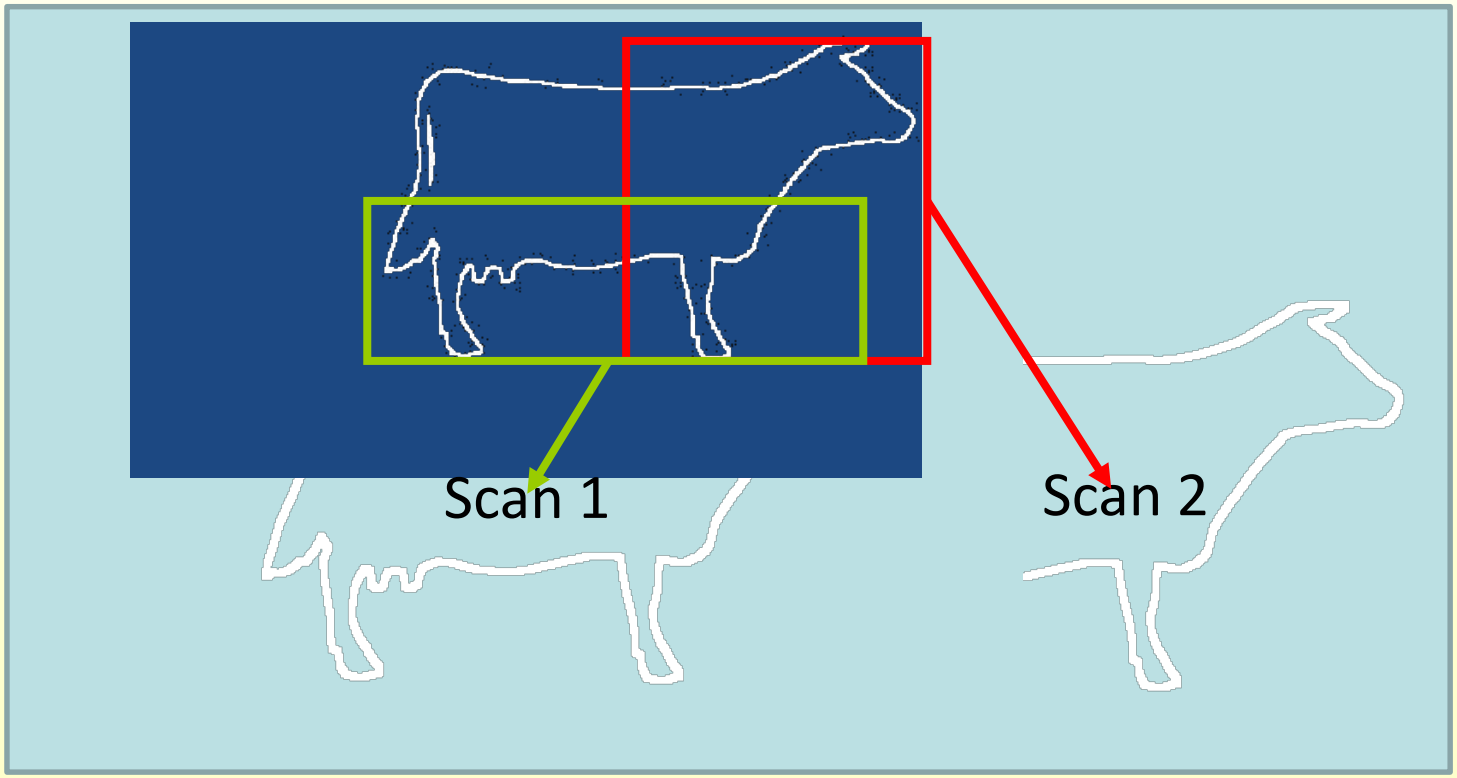
- center it about *p* (instead of the COM), and
- restrict the extent to a small region about *p*.



Shape histograms as local shape descriptors

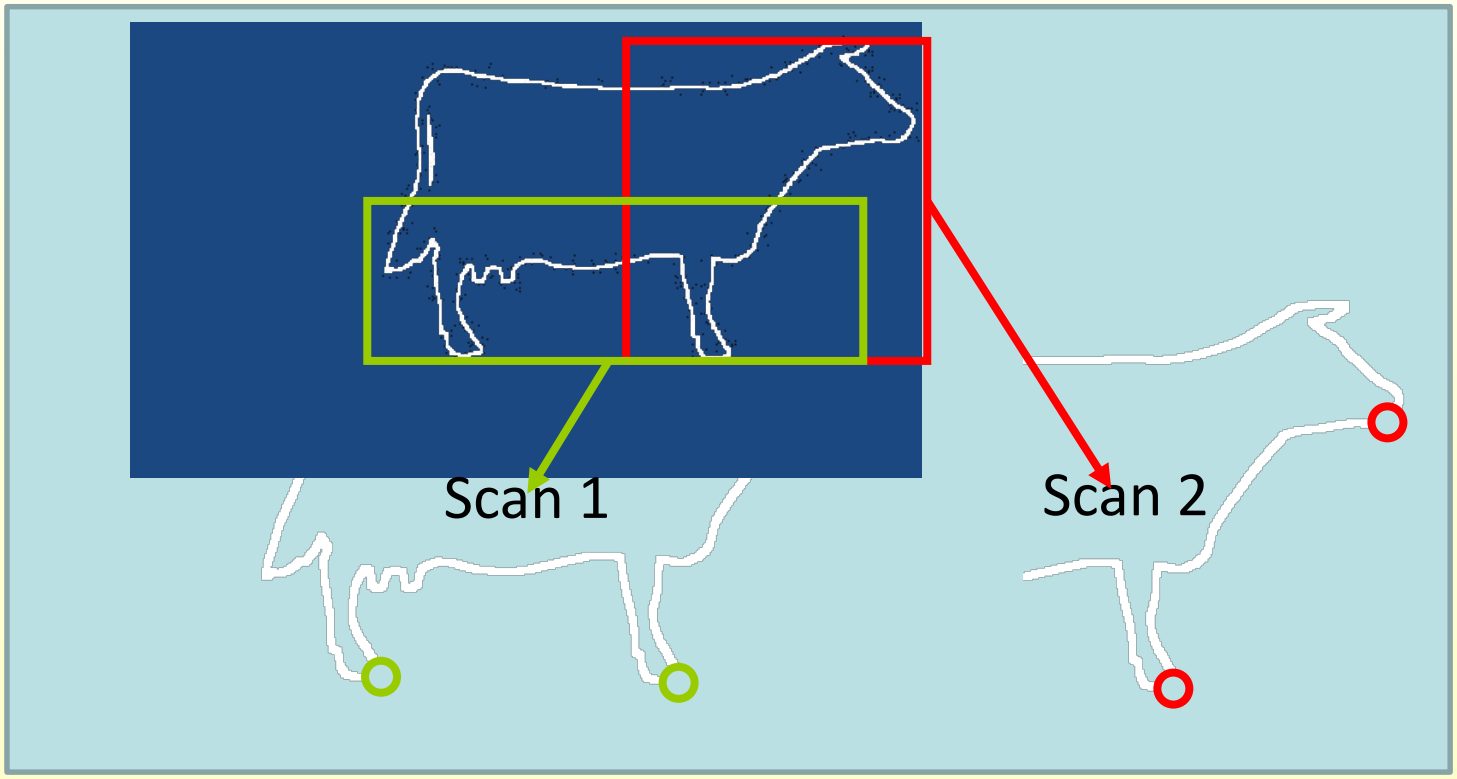
From Global to Local

Given scans of a model:



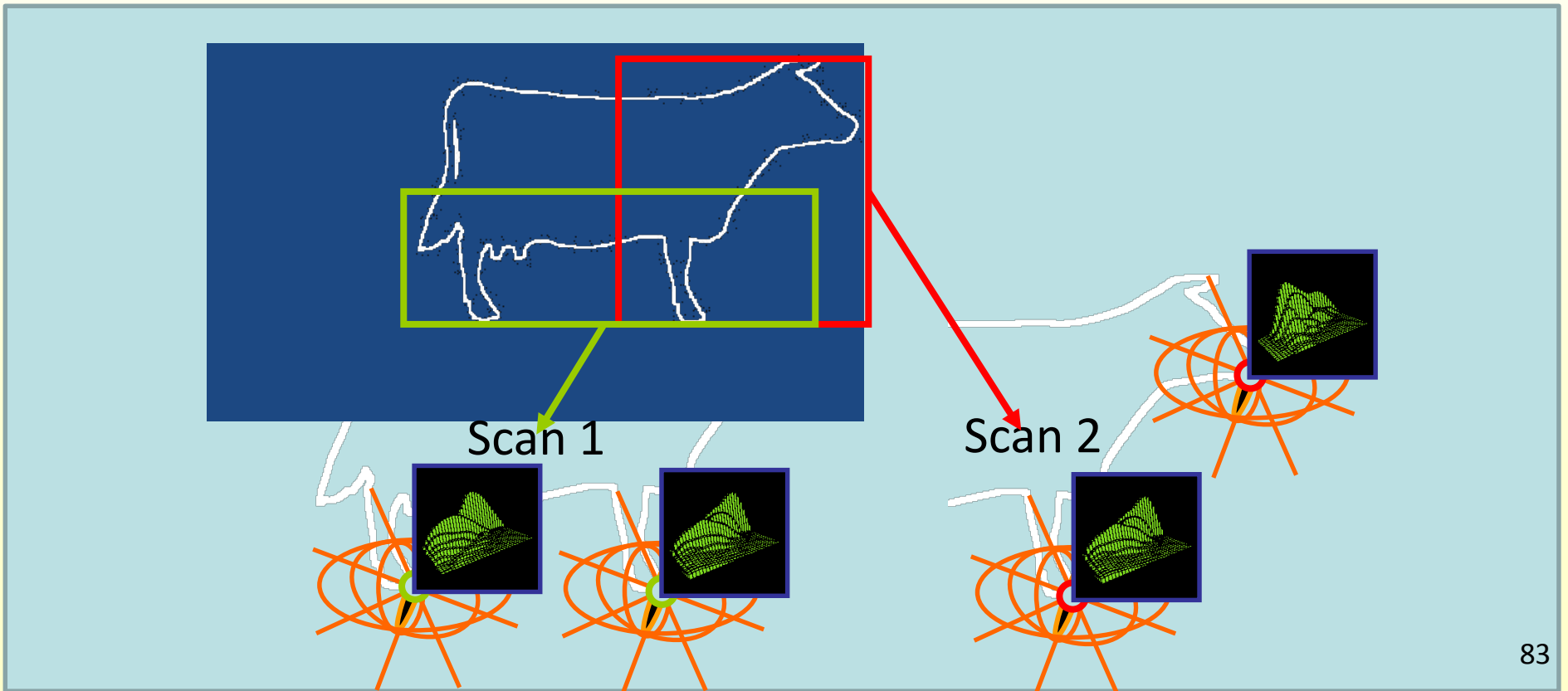
From Global to Local

- Identify the feature points



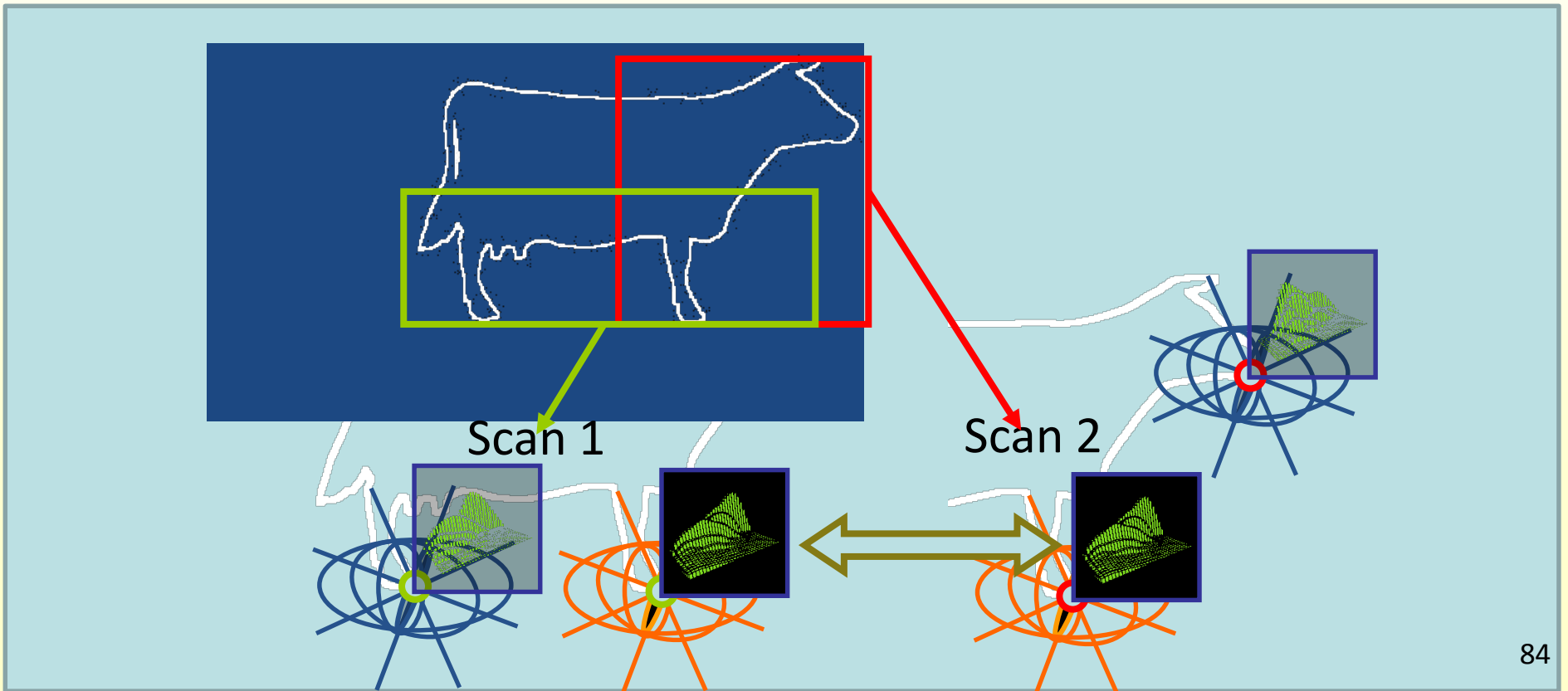
From Global to Local

- Identify the features points
- Compute a local descriptor for each feature



From Global to Local

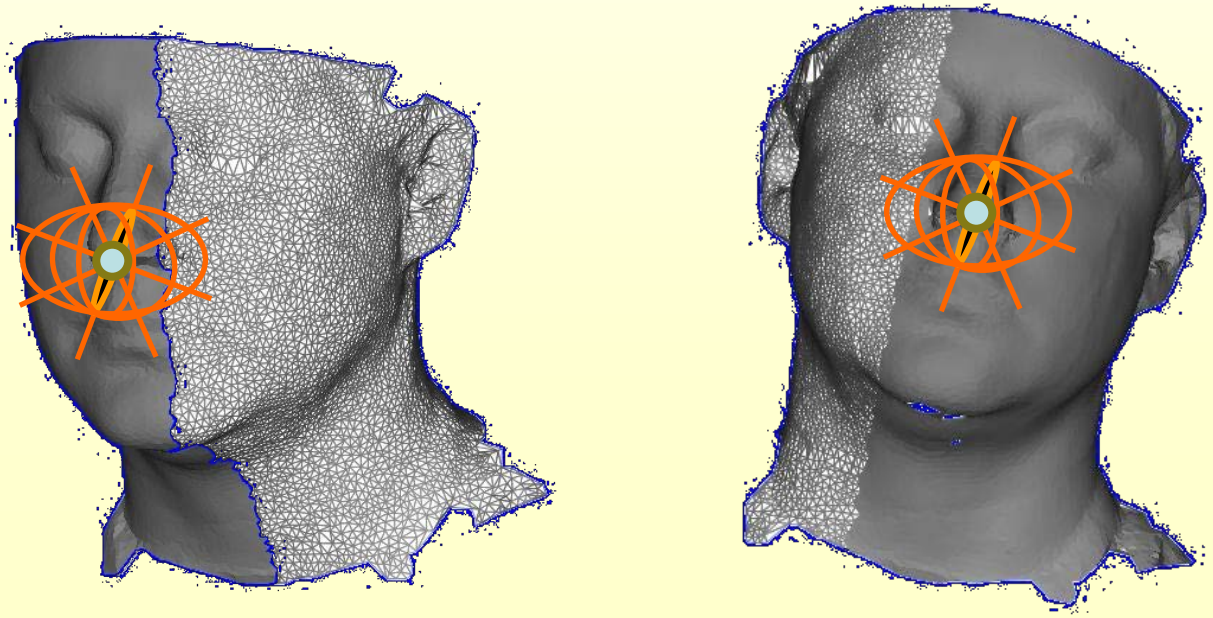
- Identify the features
- Compute a local descriptor for each feature
- For features correspond → descriptors are similar



Pose Normalization

From Global to Local

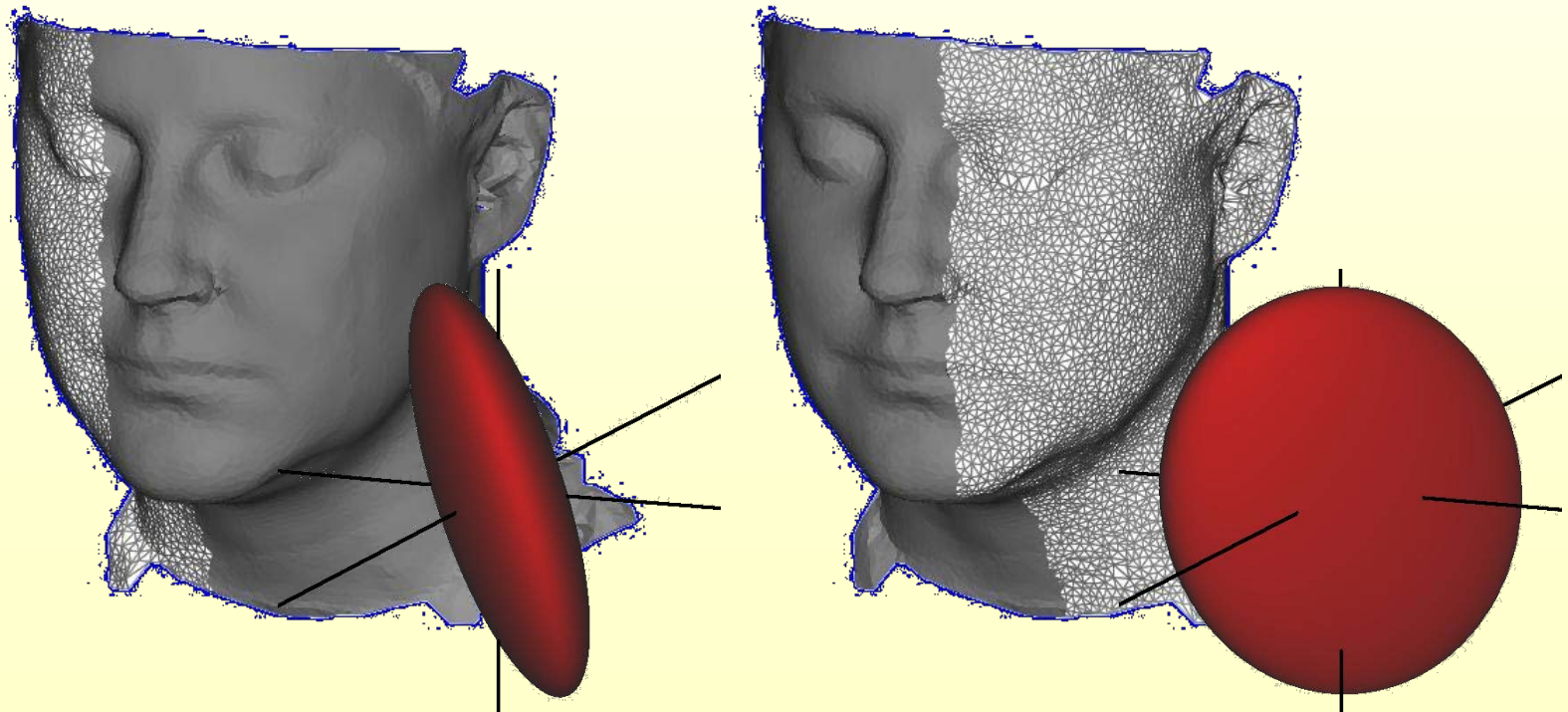
- ✓ Translation: Accounted for by centering the descriptor at the point of interest.
- ✗ Rotation: We still need to be able to match descriptors across different rotations.



Pose Normalization

Challenge

- Since only parts of the models are given, we cannot use global normalization to align the local descriptors



Pose Normalization

Challenge

- Since only parts of the models are given, we cannot use global normalization to align the local descriptors

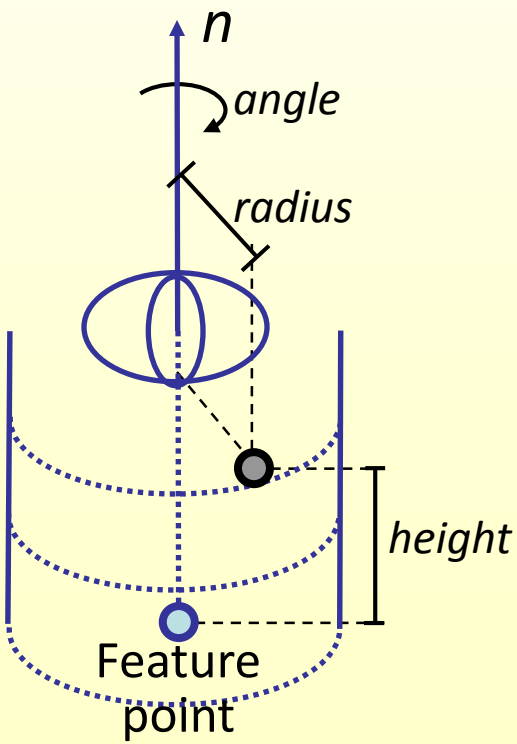
Solutions

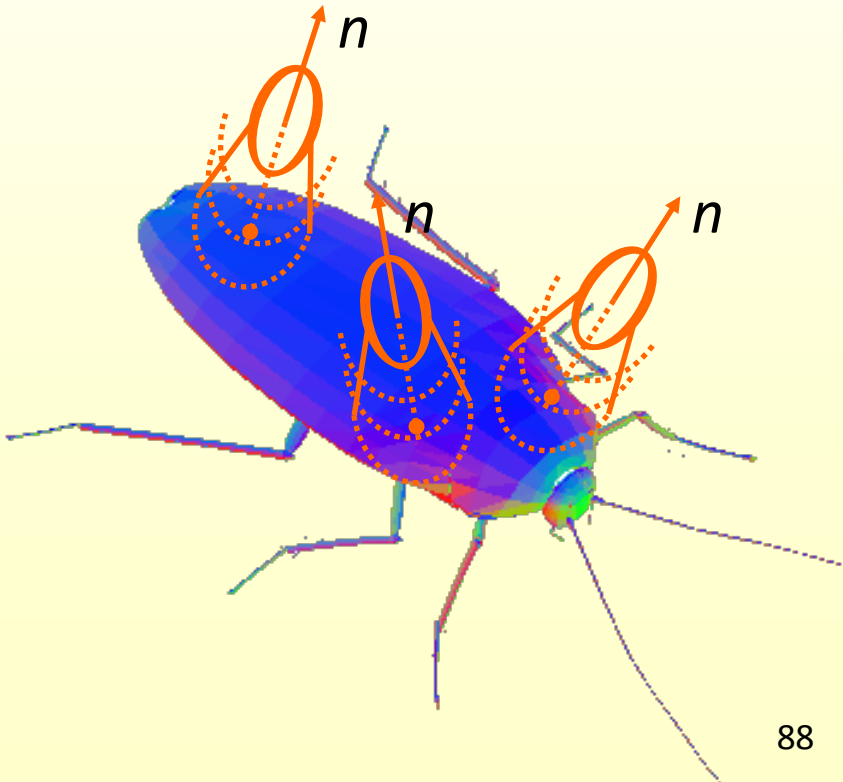
- Normalize using local information

Local Descriptors: Examples

Variations of Shape Histograms:

For each feature, represent its local geometry in cylindrical coordinates about the normal



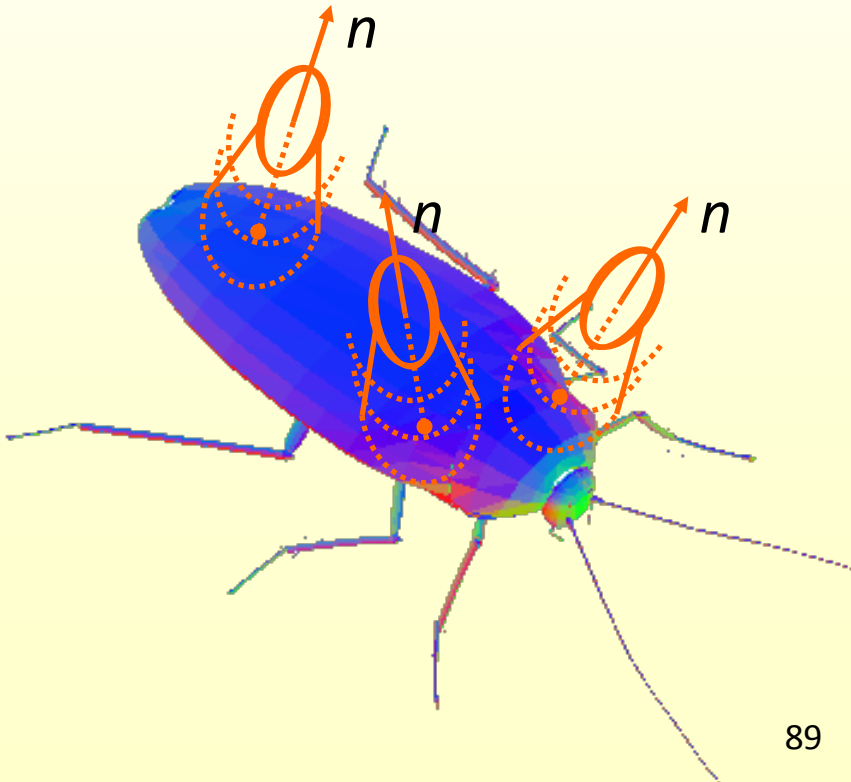


Local Descriptors: Examples

Variations of Shape Histograms:

For each feature, represent its local geometry in cylindrical coordinates about the normal

- ***Spin Images***: Store energy in each normal ring

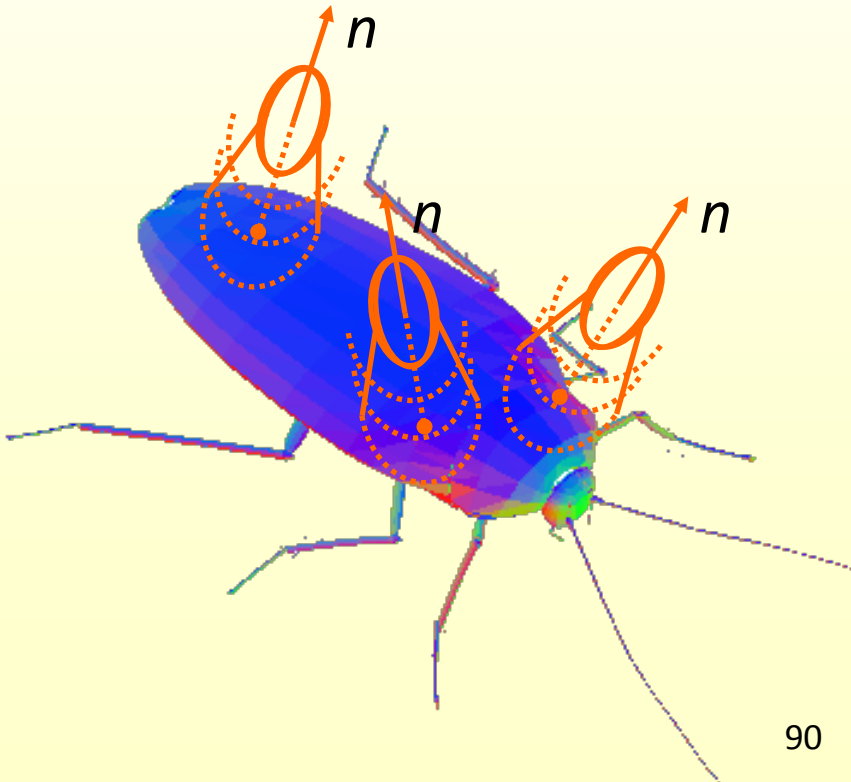


Local Descriptors: Examples

Variations of Shape Histograms:

For each feature, represent its local geometry in cylindrical coordinates about the normal

- ***Spin Images***: Store energy in each normal ring
- ***Harmonic Shape Contexts***: Store power spectrum of each normal ring

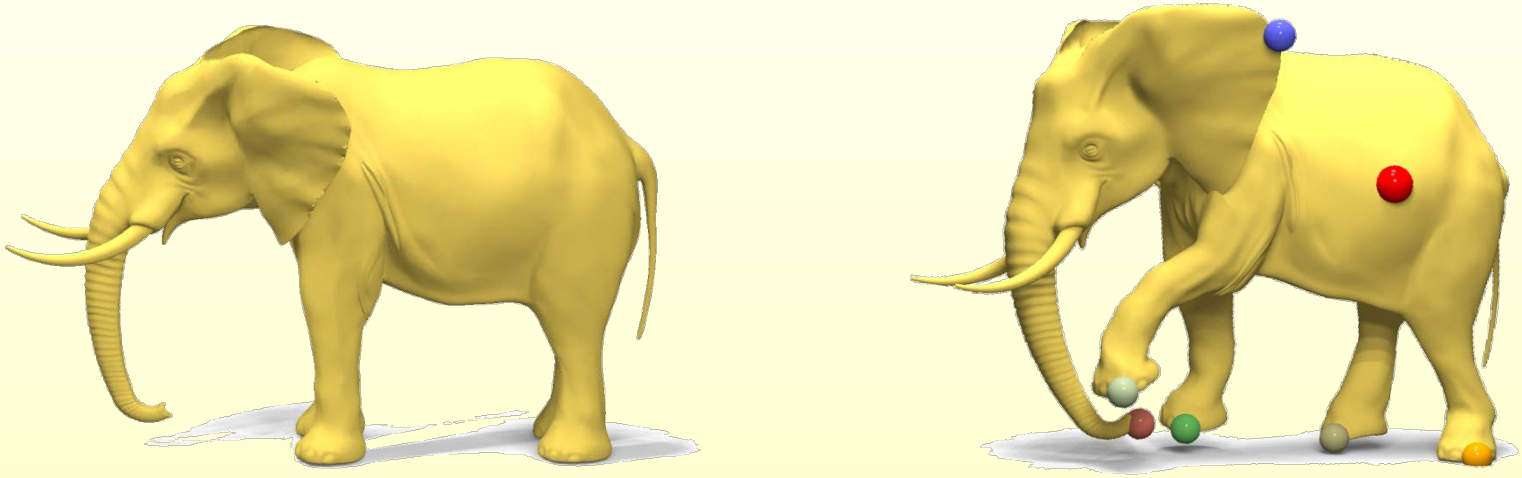


Intrinsic Descriptors Deformable Matching

Shape Matching

Problem:

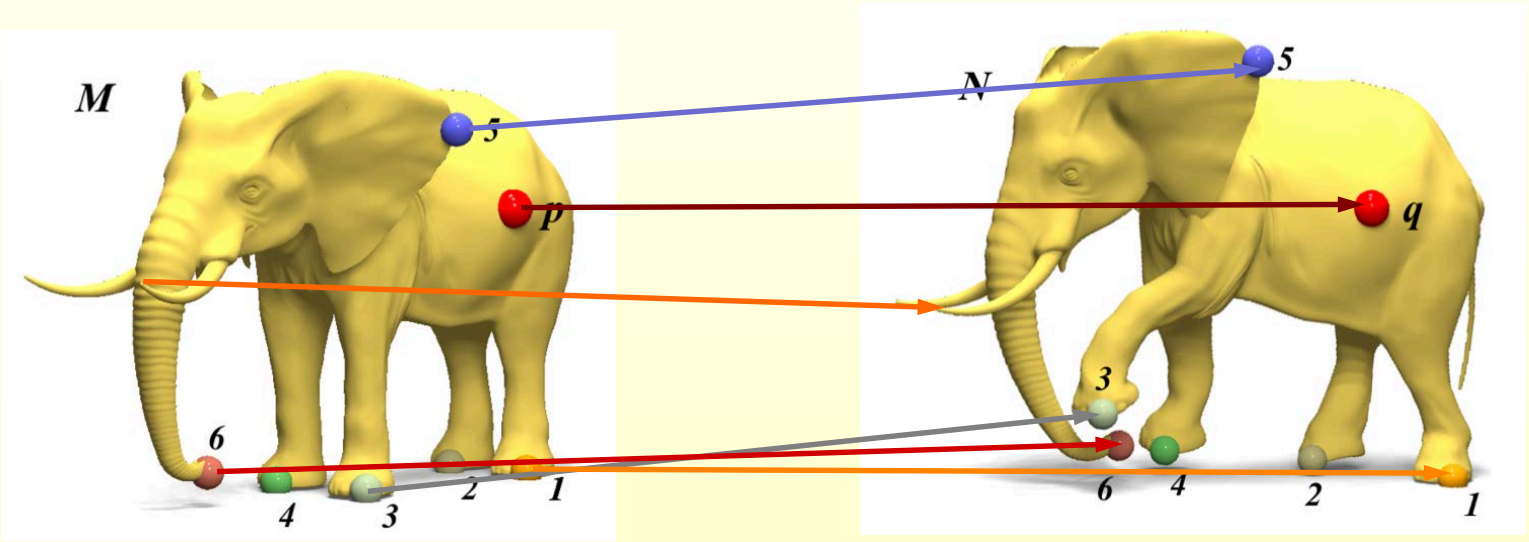
Given a pair of shapes, find corresponding points.



Shape Matching

Problem:

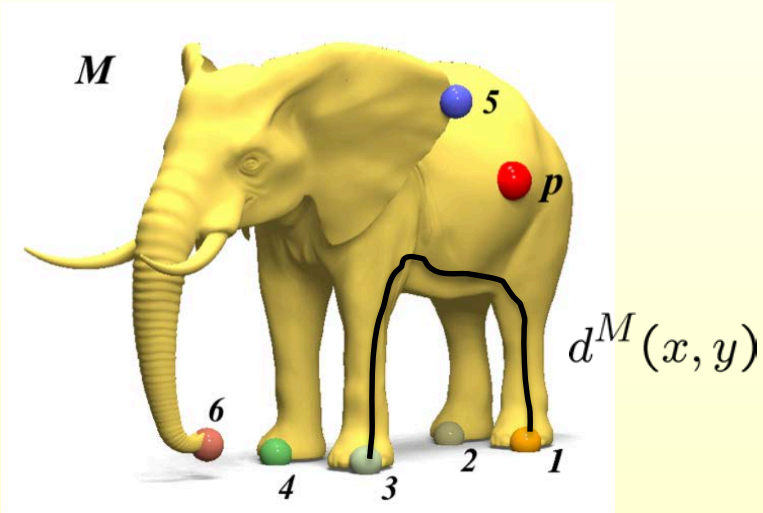
Given a pair of shapes, find corresponding points.

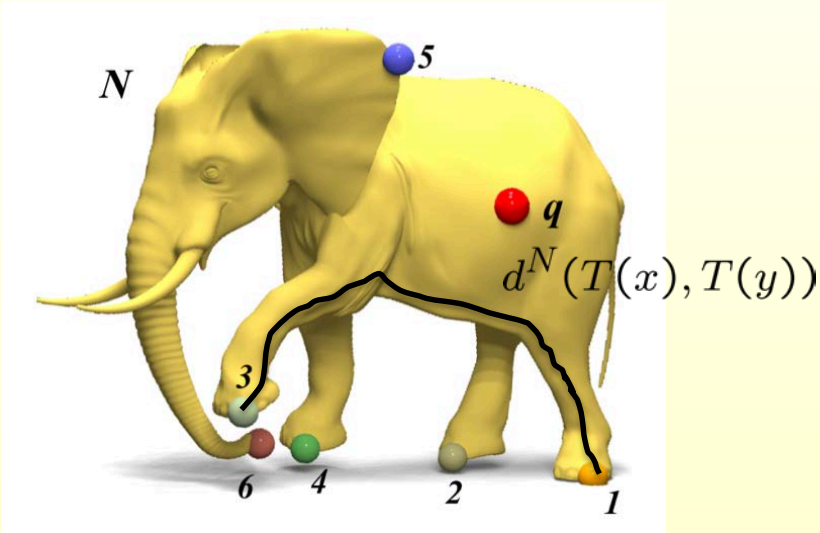


Non-Rigid Shape Matching

Problem:

Given a pair of shapes, find corresponding points.





Find a correspondence that preserves intrinsic (geodesic) distances on the shapes.

Gromov-Hausdorff: Intrinsic Measures of Shape Similarity

- Gromov-Hausdorff distance: a second order optimization over mappings or correspondences

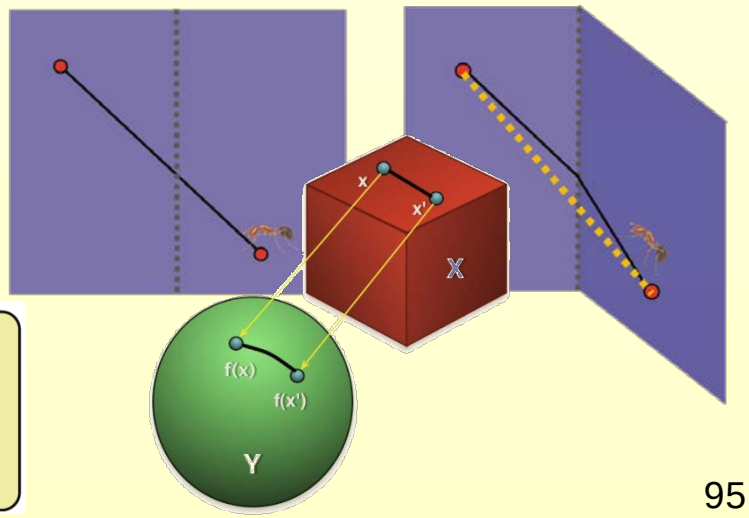
$$\Gamma_{X,Y}(x, y, x', y') := \left| d_X(x, x') - d_Y(y, y') \right|$$

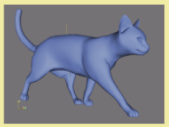
intrinsic distance distortion

$$d_{\mathcal{GH}}(X, Y) = \frac{1}{2} \inf_R \|\Gamma_{X,Y}\|_{L^\infty(R \times R)}$$

evaluated via intrinsic distances

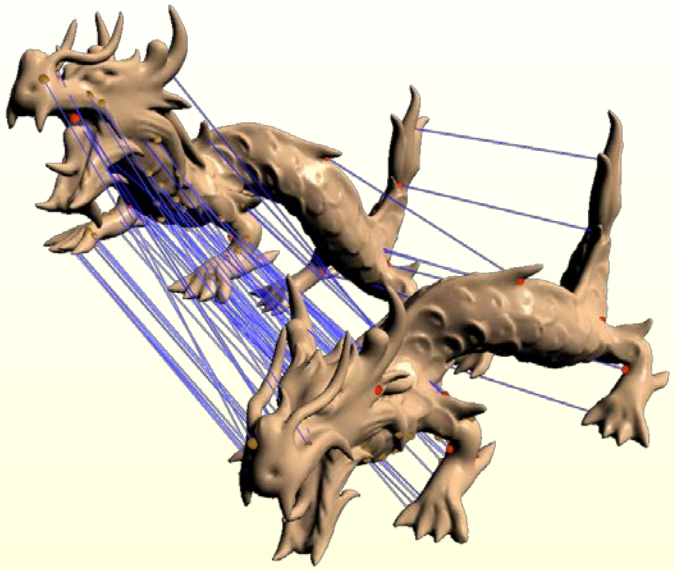
This is a minimization over all correspondences!

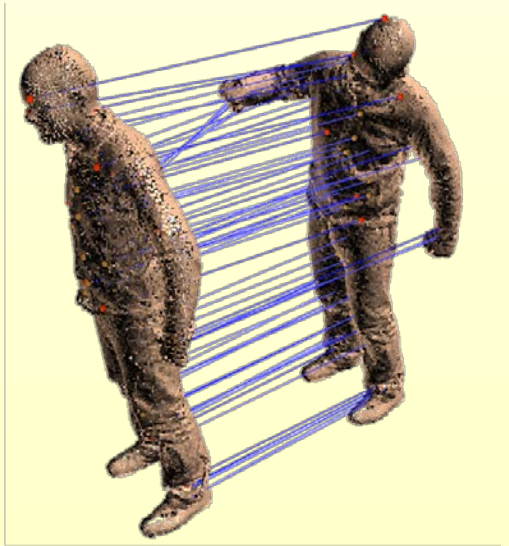


$\mathbf{dist}($  ,  $) = 0$?

Near Isometries Preserve Local Structure

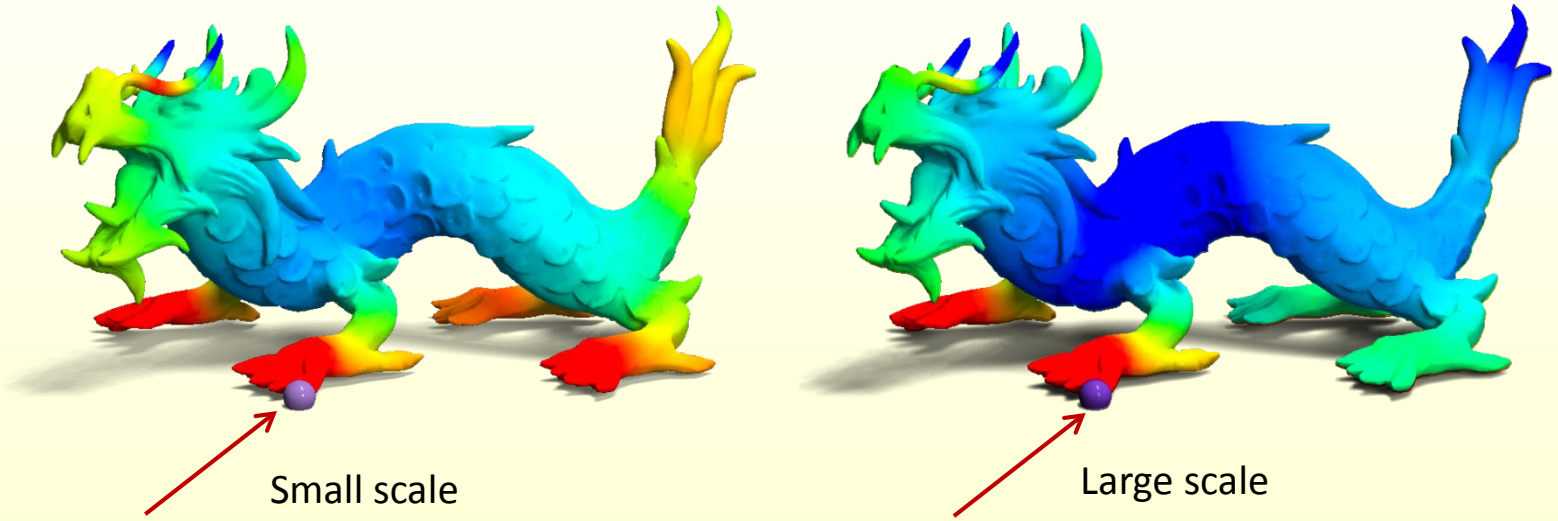
- Optimal alignment can be defined in terms of certain intrinsic but hard-to-compute shape distances, such as Gromov-Hausdorff
- We are mostly interested in the near isometric case
- Can we define certain descriptors of point neighborhoods which can guide us at establishing good mappings?
- What should be the scale of these neighborhoods?





The Issue of Scale

- Given a point (●) on a shape, find other points with “similar” neighborhoods



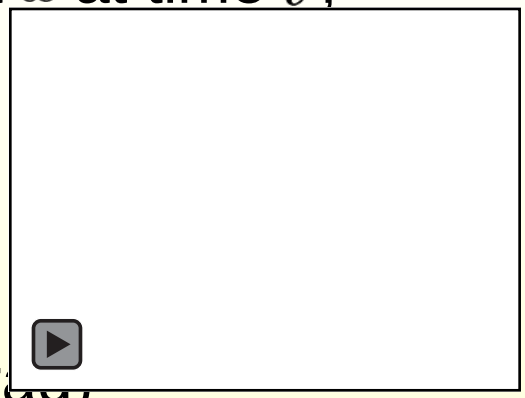
- Inherently multiscale question: on a manifold, locally all points are the same. Need a meaningful way to compare point neighborhoods at different scales
- At what scale do neighborhoods become unique?

Back to Heat Diffusion

- Heat diffusion on a Riemannian manifold:

  If $u(x,t)$ is the amount of heat at point x at time t, then

$$\frac{\partial u}{\partial t} = \Delta u$$



  Δ : Laplace-Beltrami Operator (div grad)

- Given an initial distribution $f(x)$. After time t:

$$f(x,t) = e^{-t\Delta} f$$

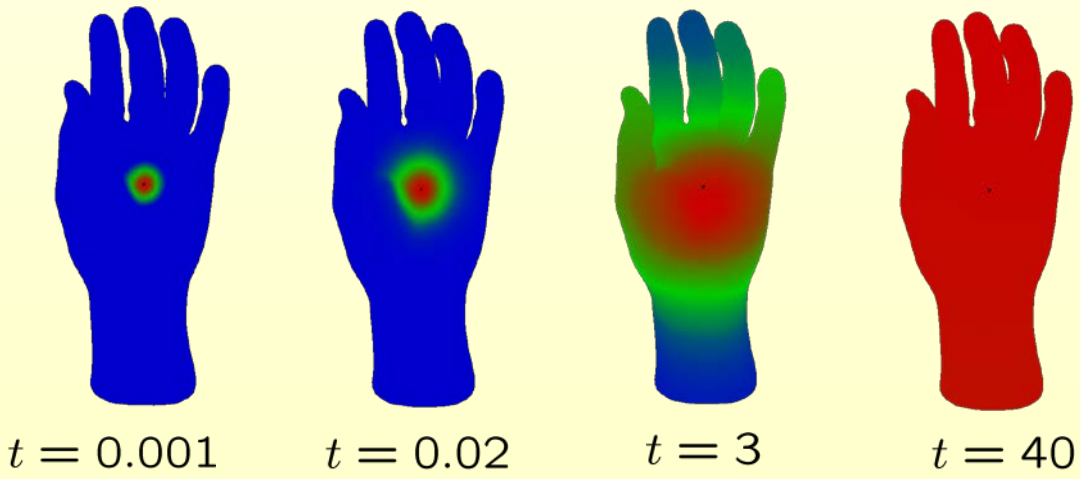
H_t heat operator

The Heat Kernel

- Heat kernel $k_t(x, y)$

$$f(x,t) = \int_{\mathcal{M}} k_t(x,y) f(y) dy$$

$k_t(x, y)$: amount of heat transferred from x to y in time t.
How well x and y are connected at scale t



Background

- Heat Kernel $k_t(x, y)$. Also the probability density function of Brownian motion on $\mathcal{M}$:

$$\mathbb{P}\left(W_x^t \in C\right) = \int_C k_t(x, y)dy$$

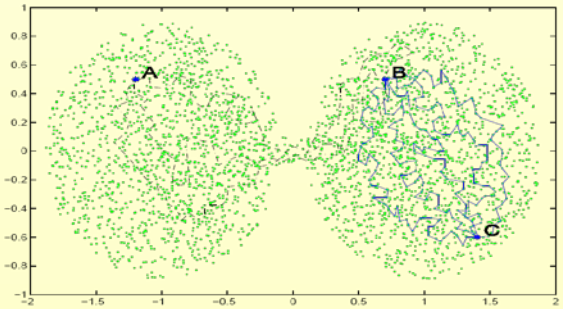
- Intuitively: weighted average over all paths possible between x and y in time t

- Related to Diffusion Distance:

$$D_t(x, y) = k_t(x, x) - 2k_t(x, y) + k_t(y, y)$$

a robust multi-scale measure

of proximity



Heat Kernel Properties

Basic Properties

- $k_t(x,y) = k_t(y,x)$
- $k_{t+s}(x,y) = \int_M k_t(x,z)k_s(z,y)dz$
- $k_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$

Eigenfunctions of LB

Heat Kernel Properties

- Invariant under isometric deformations

  If $T : X \to Y$ is an isometry, then:

  $$k_t(X, Y) = k_t(T(x), T(y))$$

- Conversely: it characterizes the shape up to isometry.

  If $k_t(X, Y) = k_t\left(T(x), T(y)\right) \ \forall \ x, y, t$ then:

  T is an isometry.

This is because:

$$\lim_{t \downarrow 0} \left(t \log k_t(x, y)\right) = -\frac{1}{4} d^2_{\mathcal{M}}(x, y) \ \forall \ x, y$$

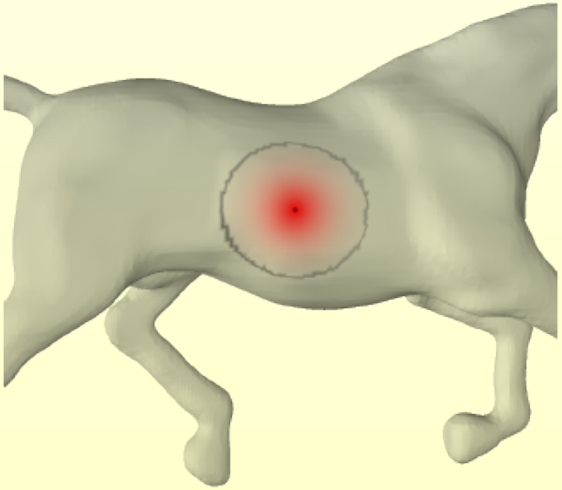
where $d_{\mathcal{M}}(\cdot, \cdot)$ is the geodesic distance

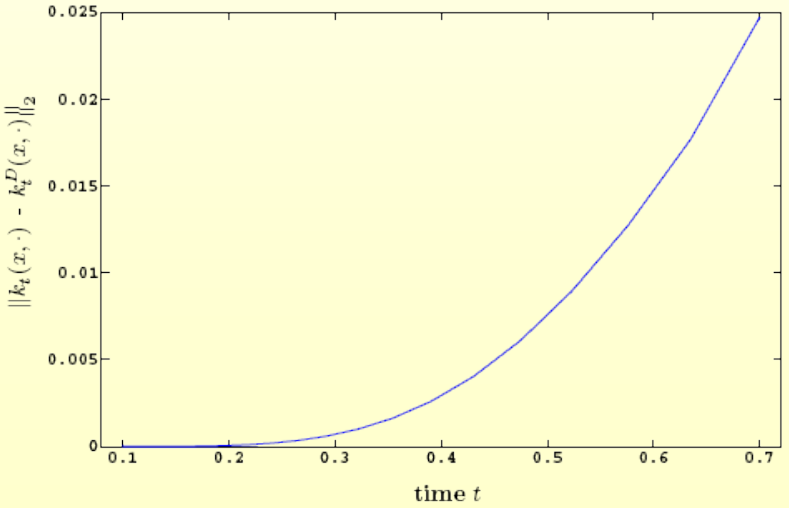
Heat Kernel Properties

- Multiscale:

  For a fixed x, as t increases, heat diffuses to larger and larger neighborhoods

  Therefore, $k_t(x, \cdot)$ is determined by (reflects the properties of) a neighborhood that grows with t

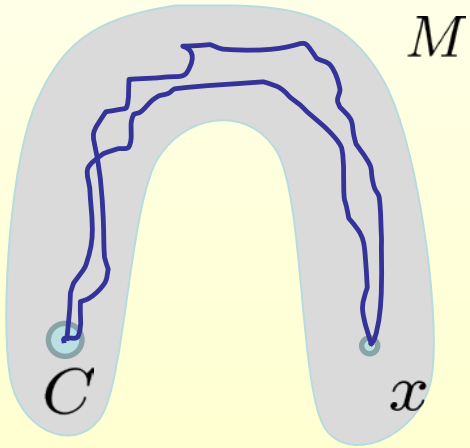




Heat Kernel Properties

- Robustness:

  $k_t(x, \cdot)$ is the probability density function of BM, a weighted average over all paths, which is generally not very sensitive to local perturbations

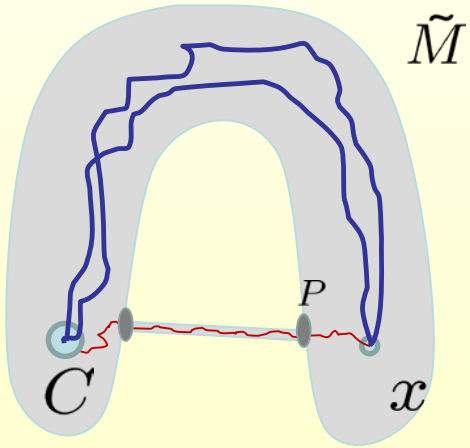


$$k_t^M(x, C) = \mathbb{P}(W_x^t \in C)$$

Heat Kernel Properties

- Robustness:

  $k_t(x, \cdot)$ is the probability density function of BM, a weighted average over all paths, which is generally not very sensitive to local perturbations



$$k_t^{\tilde{M}}(x, C) = \mathbb{P}(\tilde{W}_x^t \in C)$$

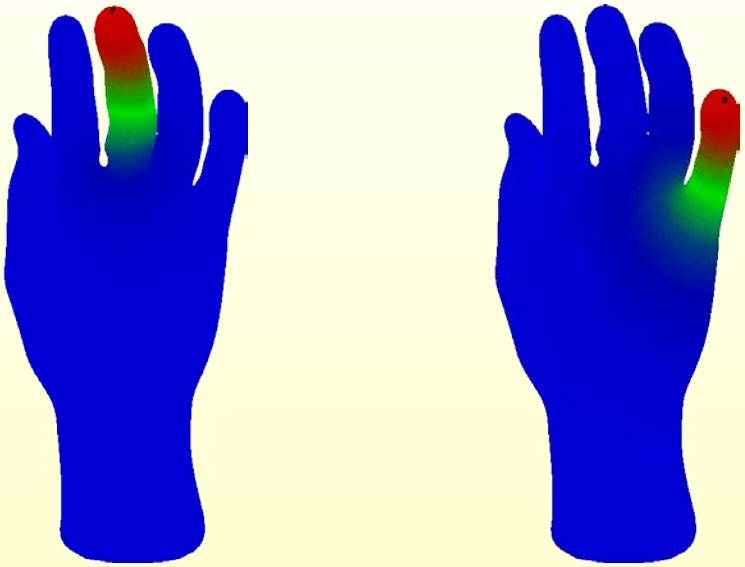
Only paths through the modified area P will change

Defining a Signature

- Let $k_t(x, \cdot)$ be the signature of x at scale t

  The heat kernel has all the properties we want

  Except easy comparison …



- $k_t(x, \cdot)$ is a function on the entire manifold
- Nontrivial to align the domains of such functions across different shapes, or even for different points of the same shape

The Heat Kernel Signature

- Let $k_t(x, \cdot)$ be the signature of x at scale t

  The heat kernel has all the properties we want.

  Except easy comparison …

- We define the Heat Kernel Signature (HKS), by restricting to the diagonal:

$$\mathsf{HKS}(x) = \{k_t(x, x), t \in \mathbb{R}^+\}$$

- Now HKSs of two points can be easily compared since they are defined on a common domain (time)

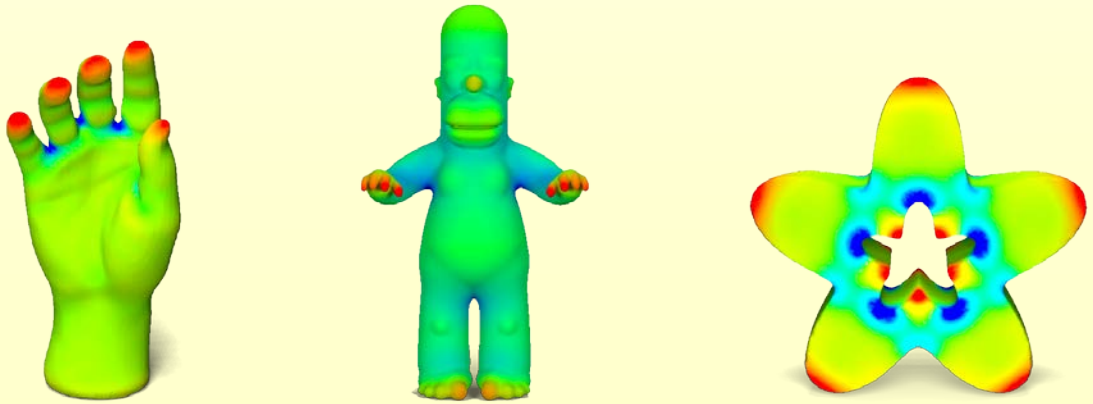
[Sun, Ovsjanikov, G., 2009]

Defining a Signature

- Since HKS is a restriction of the heat kernel, it is:
  - Robust
  - Multiscale

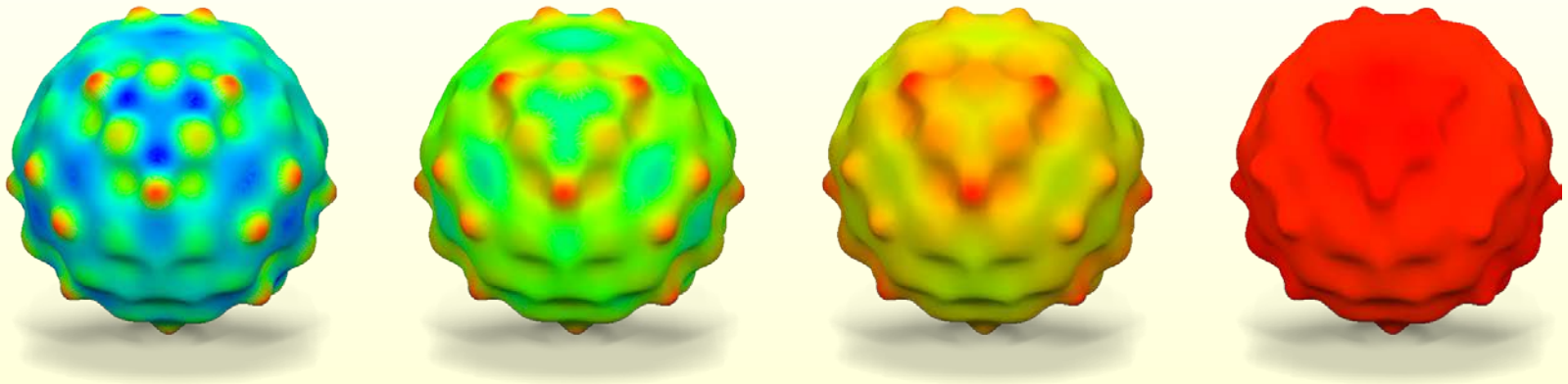
- Question: How informative is it?
  - Related to Gaussian curvature for small t :

$$k_t(x,x) = \frac{1}{4\pi t}\sum_{i=0}^{\infty} a_i t^i \qquad a_0 = 1, a_1 = \frac{1}{6}K$$



Defining a Signature

- HKS can be interpreted as a multiscale, robust, intrinsic curvature:



$t = 0.004$ $t = 0.008$ $t = 0.02$ $t = 2$

Informative Theorem

- The set of all HKSs on a shape almost always defines it up to isometry!

- Theorem: If X and Y are two compact manifolds, such that Δ_X and Δ_Y have only non-repeating eigenvalues, then a homeomorphism $T : X \rightarrow Y$ is an isometry if and only if, for all x

$$\mathsf{HKS}(x) = \mathsf{HKS}(T(x))$$

- The set of all HKSs characterizes the intrinsic structure of the manifold

Informative Theorem

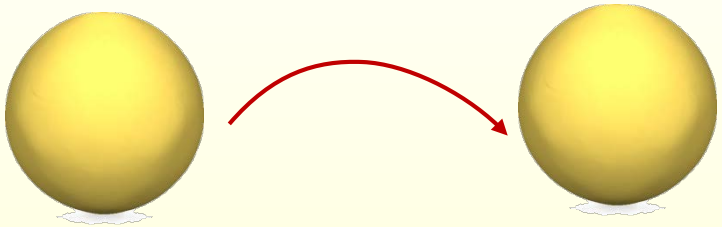
- Heat kernel is related to the eigenvalues and eigenfunctions of the LB-operator:

$$\mathrm{HKS}(x,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i^2(x)$$

- Invariant to rotations within the eigenspace

Informative Theorem

- How general is the theorem?
  - If there are repeated eigenvalues, it does not hold:



On the sphere, $\mathrm{HKS}(x) = \mathrm{HKS}(y) \ \forall \ x, y$ but there are non-isometric maps between spheres.

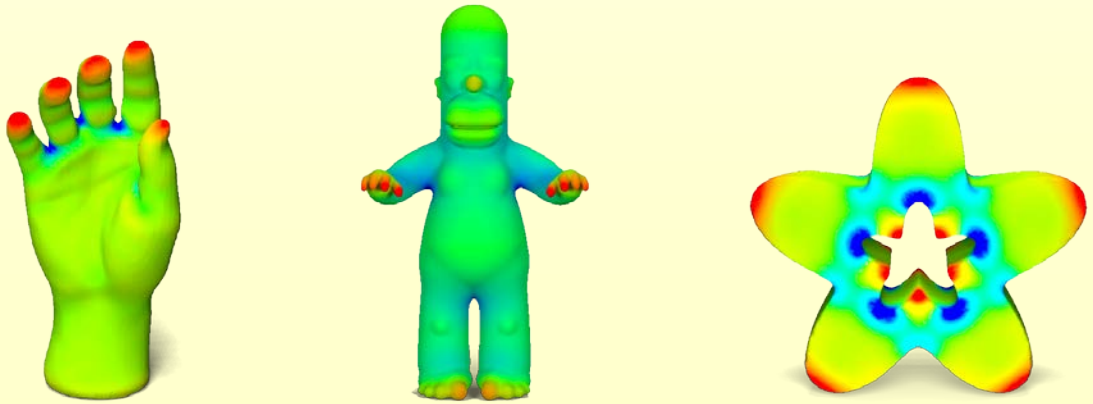
- Uhlenbeck's Theorem (1976): for "almost any" metric on a 2-manifold X, the eigenvalues of Δ_X are non-repeating

Informative Property

- Conclusion:
  - HKS is informative for individual points
  - And, as a set, for the entire shape

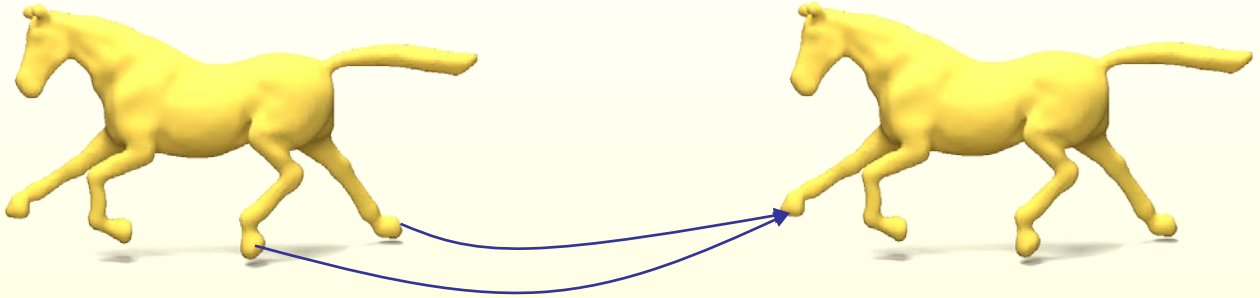
Can be used both for multiscale point matching and for shape comparison

$$\mathrm{HKS}(x) = \{k_t(x,x), t \in \mathbb{R}^+\}$$

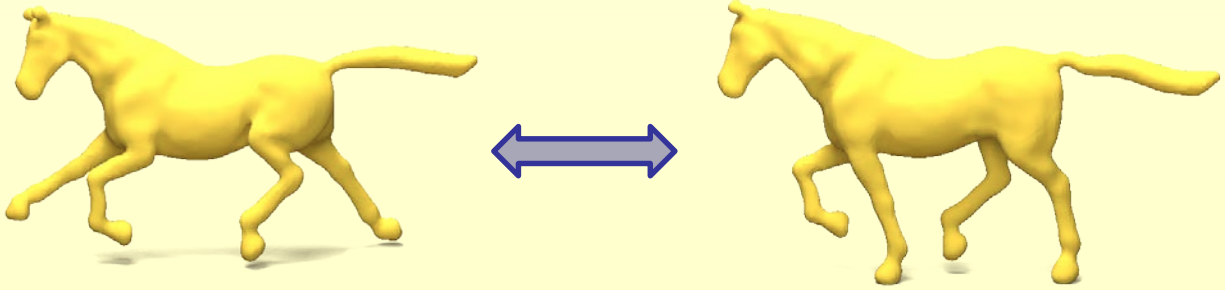


Applications

- Multi-scale matching with HKS, structure discovery



- Shape comparison
  - Shape retrieval using HKS
  - Spectral version of Gromov-Hausdorff



Multiscale Matching

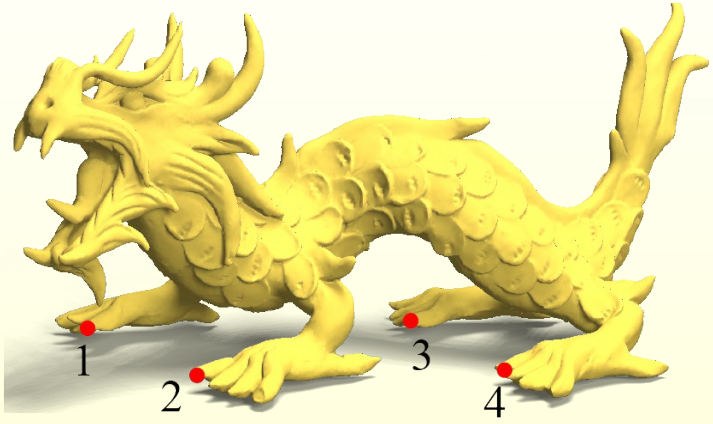
- Two heuristics for making HKSs commensurable:
  - For a fixed point x, sample HKS on a logarithmic scale at times t_i
  - For a fixed time t scale each HKS, by the sum over all points of *M*

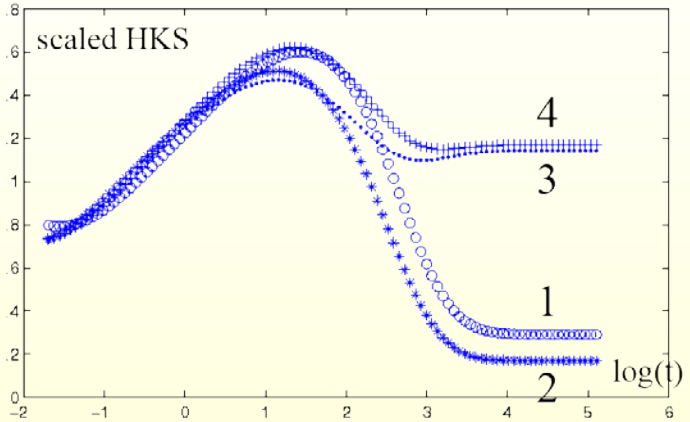
$$\text{HKS}(x) = \left\{ \frac{k_{t_i}(x,x)}{\sum_j e^{-t_i \lambda_j}}, i \in 1, 2, .., 100 \right\}$$
$$t_i = \alpha^i t_0$$

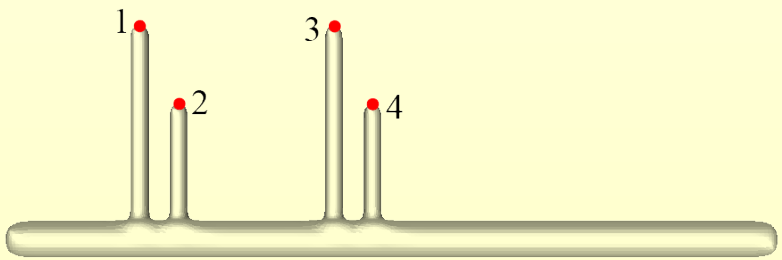
- Compare using L2 norm of the HKS vectors.

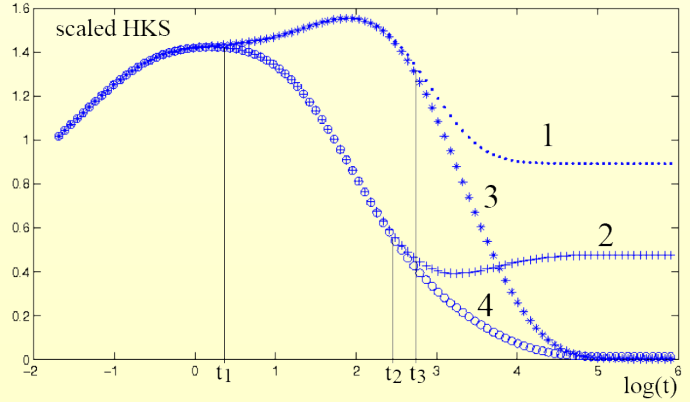
Multiscale Matching

- Comparing points through their HKS signatures:





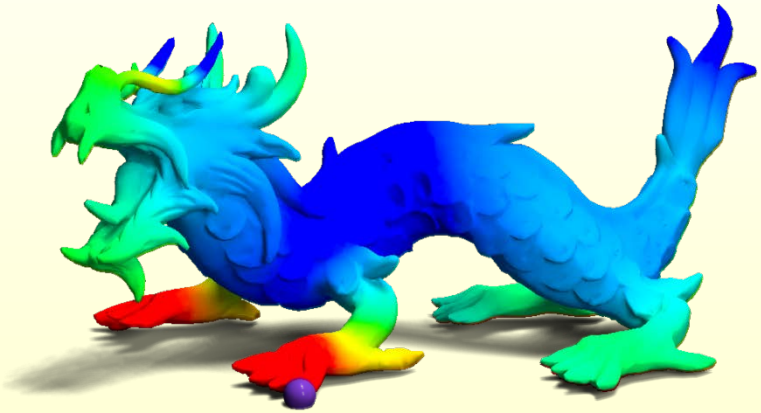




Multiscale Matching

- Comparing points through their HKS signatures:

Medium scale



Full scale

Multiscale Matching

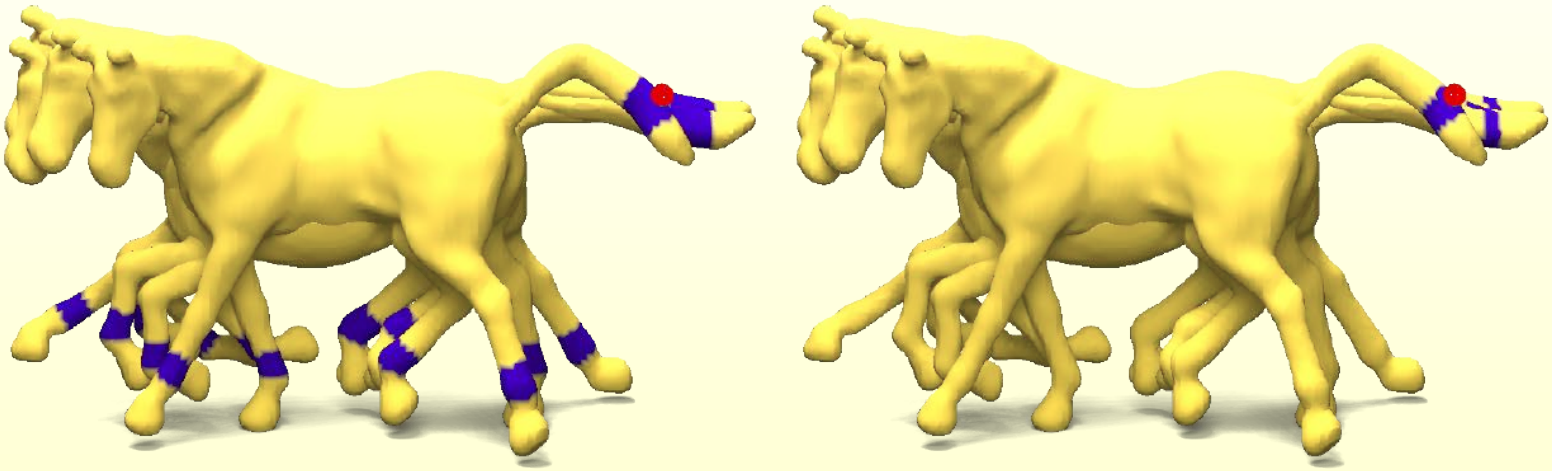
- Finding similar points – robustly:

Medium scale

Full scale

Multiscale Matching

- Finding similar points across multiple shapes:

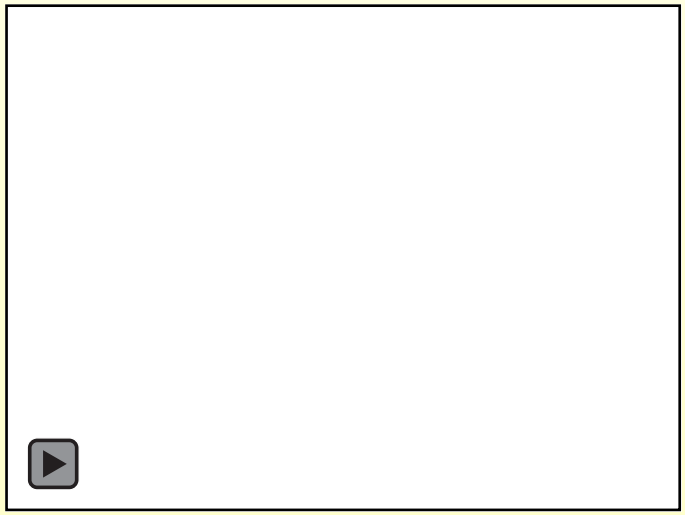


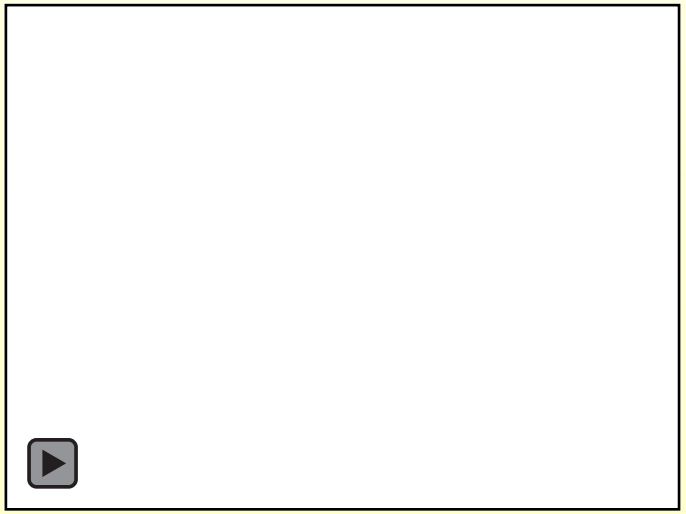
Medium scale

Full scale

Feature Detection

- Persistent feature detection:
  - Intuition: heat diffuses slower at points with high curvature. Heat will tend to concentrate in “hot spots” – extremities of the surface
  - Approach: track the local maximum of the heat kernel for increasing t

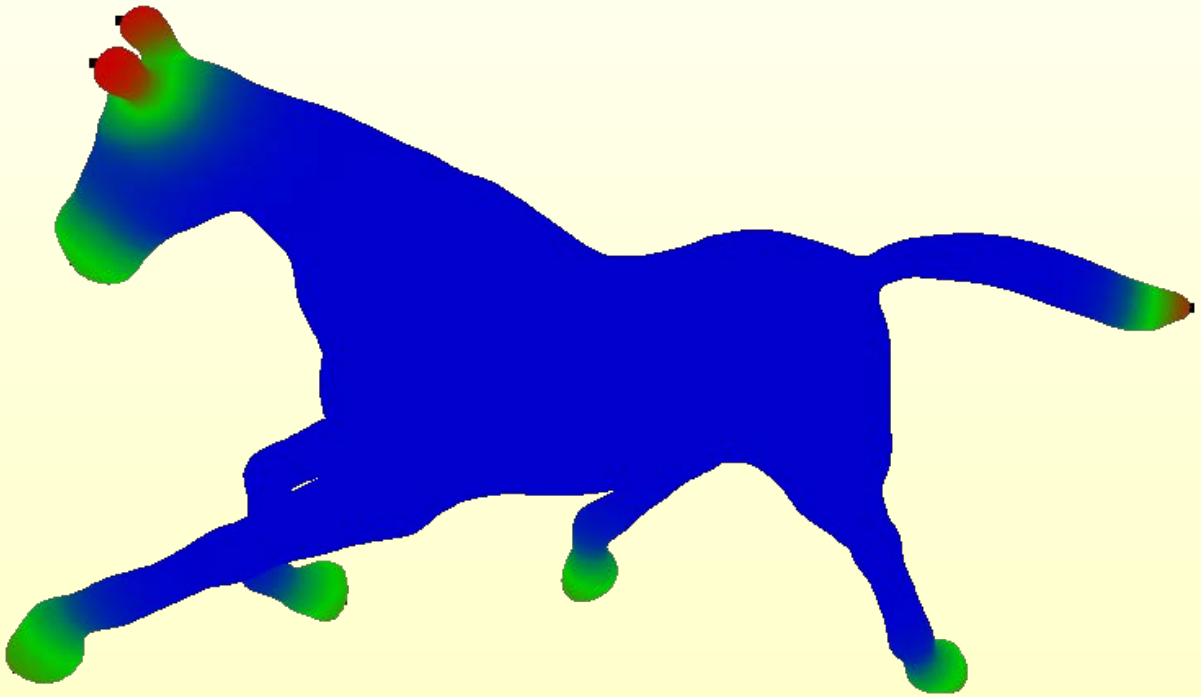




Feature Detection

- Persistent feature detection:
  - Find points that are long term maxima of their heat kernels:

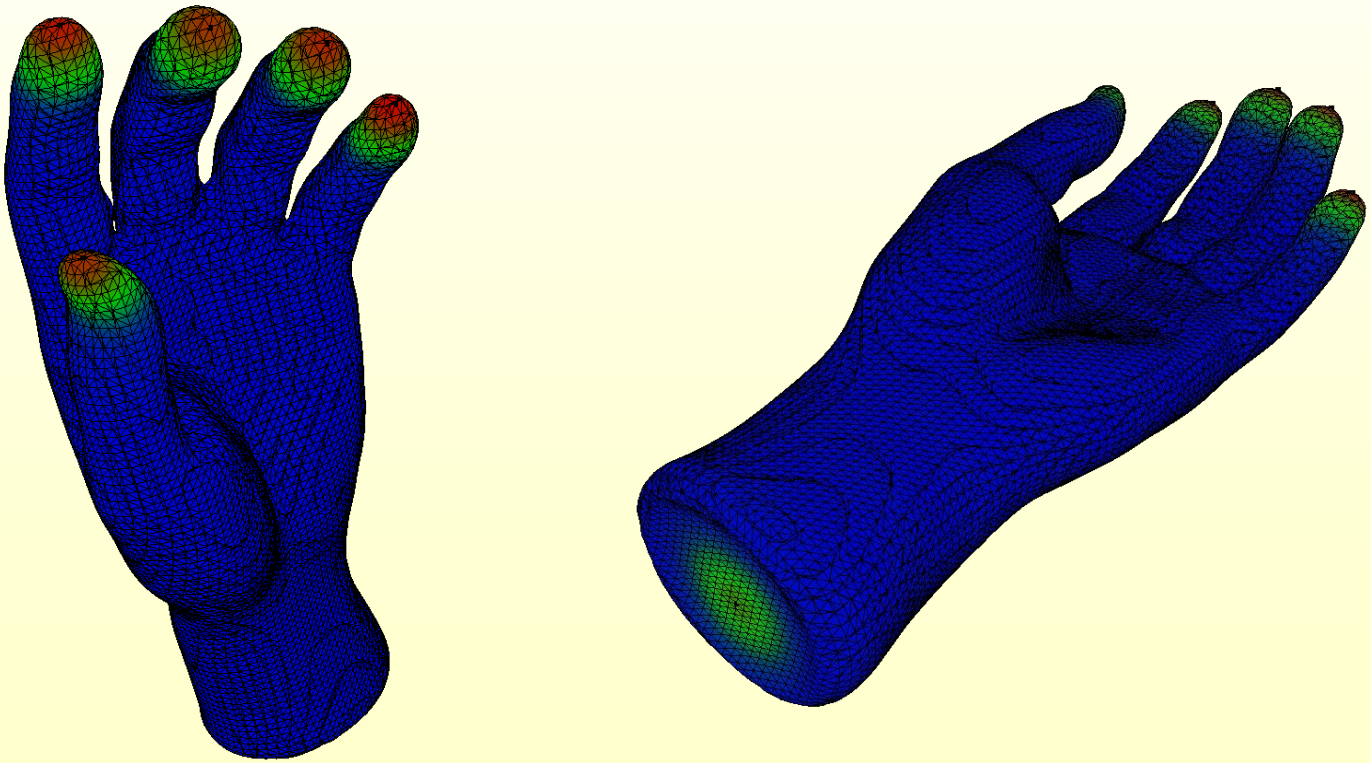
$$k_t(x, \cdot)$$



Feature Detection

- Persistent feature detection:
  - Find points that are long term maxima of their heat kernels:

$$k_t(x, \cdot)$$



Feature Detection

- Persistent feature detection:
  - Find points that are long term maxima of their heat kernels:

$$k_t(x, \cdot)$$

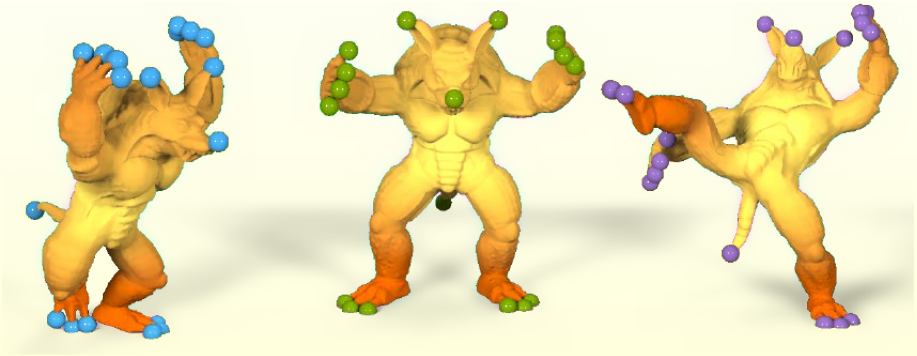
  - This may be expensive since the heat kernel at every point is a function over the whole shape. However, long term behavior at nearby points is similar due to mixing
  - Approximation: find points that are local maxima of

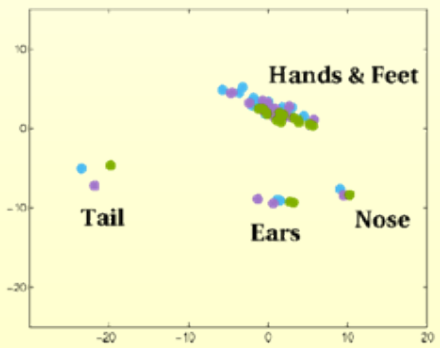
$$k_t(x, x)$$

    for large enough t

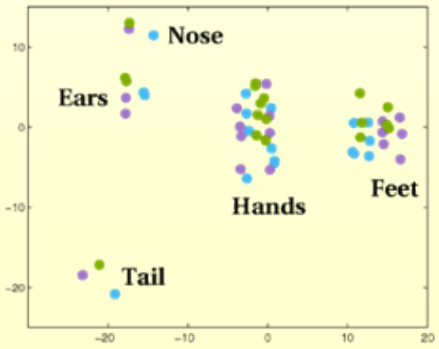
Shared Structure

- 2D MDS embedding of feature points on three shapes according to distances of their HKS





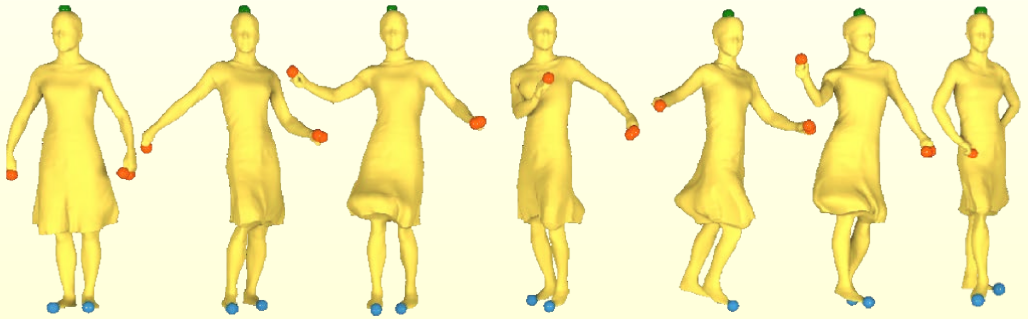
$t = [0.1, 4]$



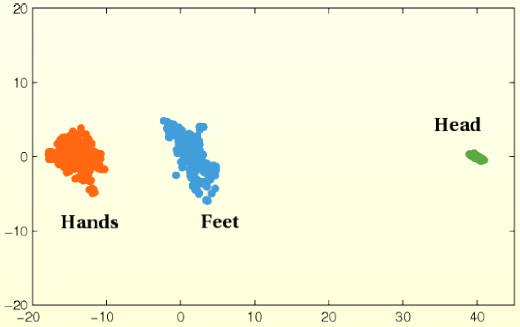
$t = [0.1, 80]$

Shared Structure

- 2D MDS embedding of feature points on 175 shapes according to distances of their HKS.

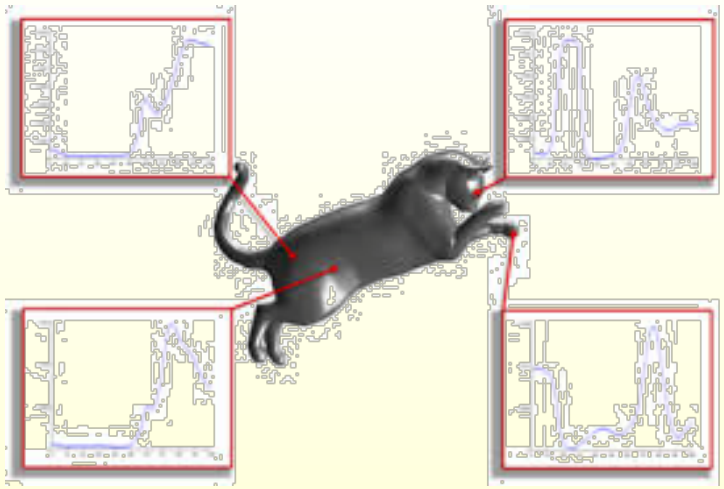


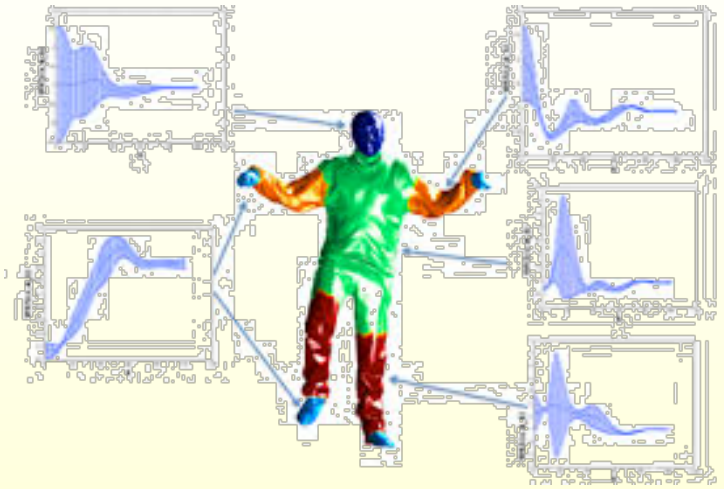
Feature points found on a few poses of the dancer model by Vlasic *et al.*



MDS of features from all 175 poses using a full range of scales

The Wave Kernel Signature (WKS)



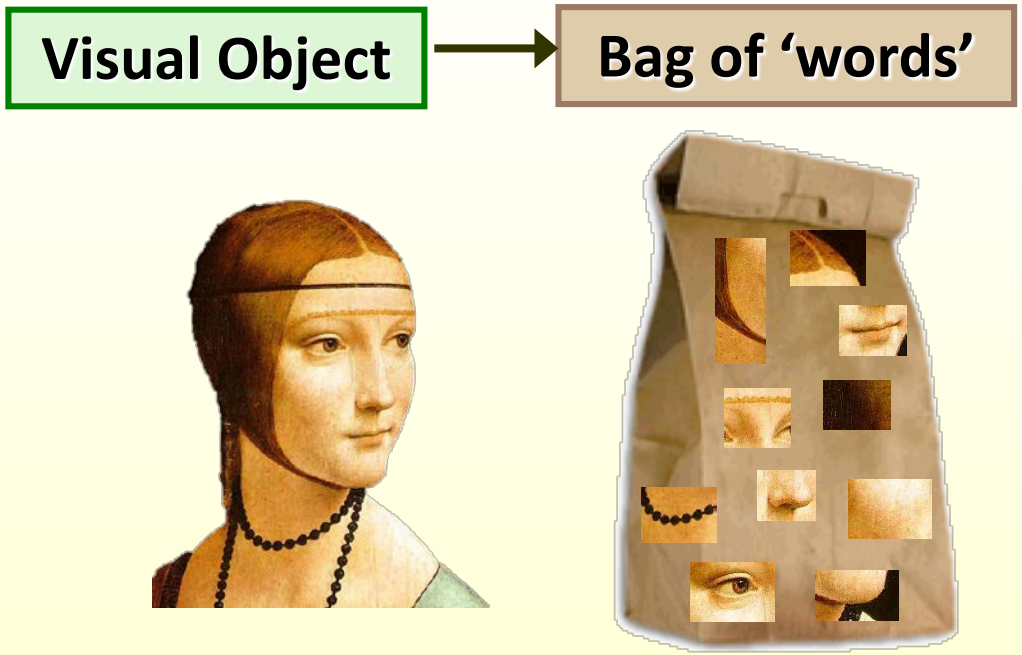


Based on solutions of the quantum Schrödinger equation

$$\frac{\partial \psi}{\partial t}(x,t) = i\Delta\psi(x,t)$$

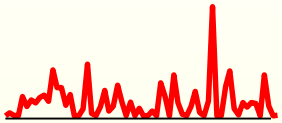
Shape Search

Bag-of-Words Models

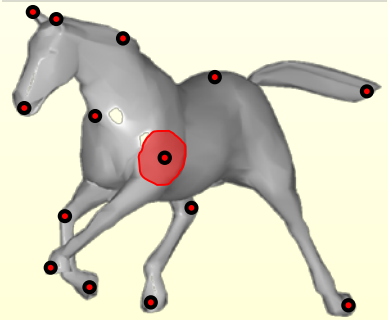


Spatial Words

Geometric words

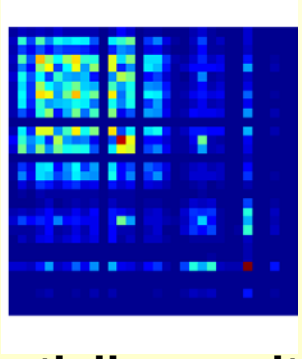


Bag of geometric words



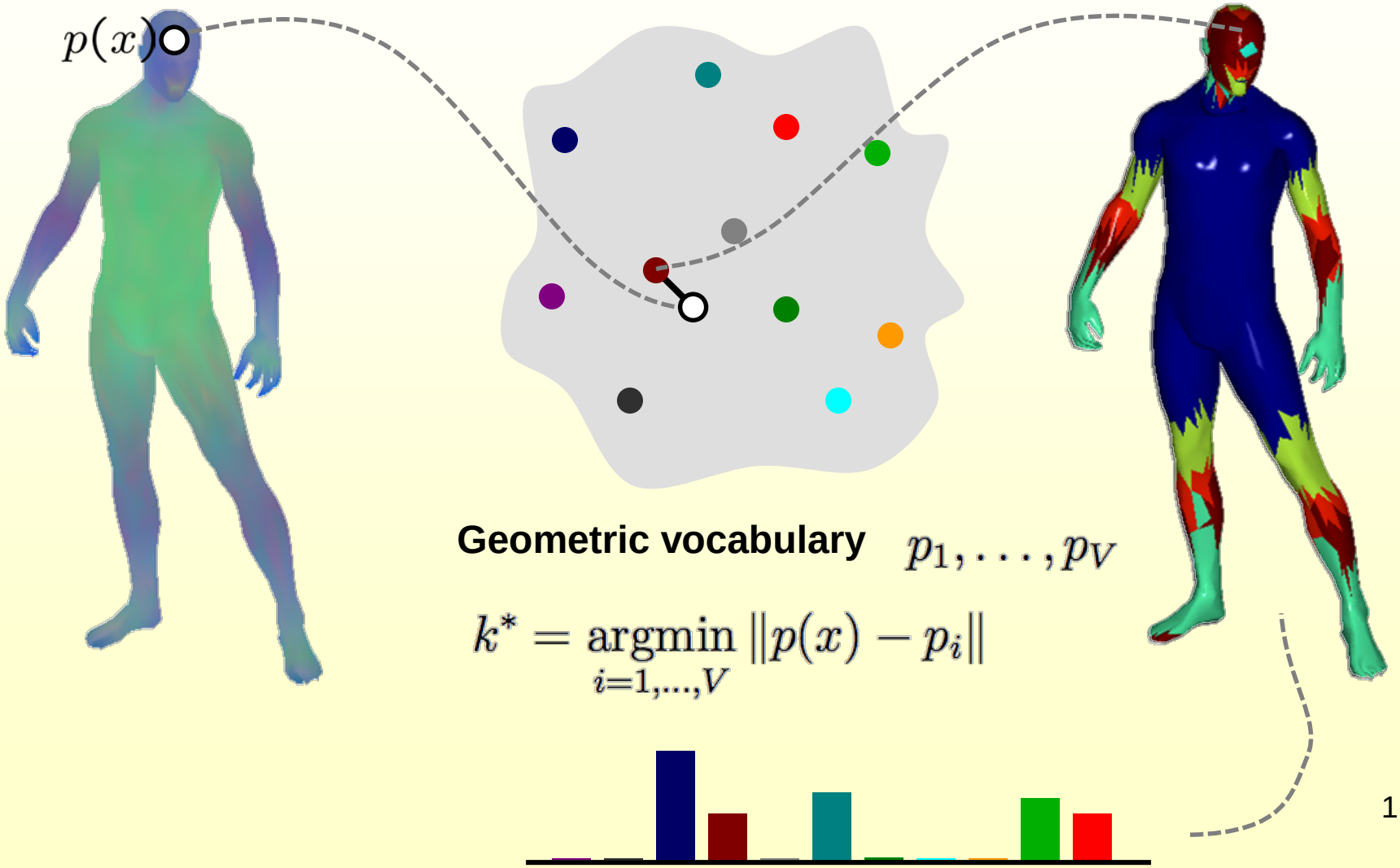
Feature descriptor

Geometric expressions

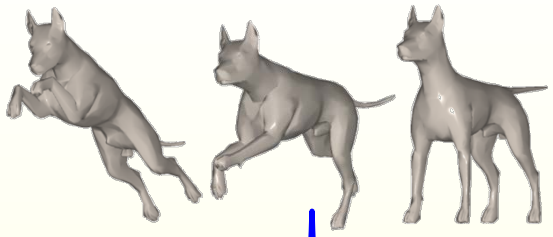


Spatially-sensitive bag of words

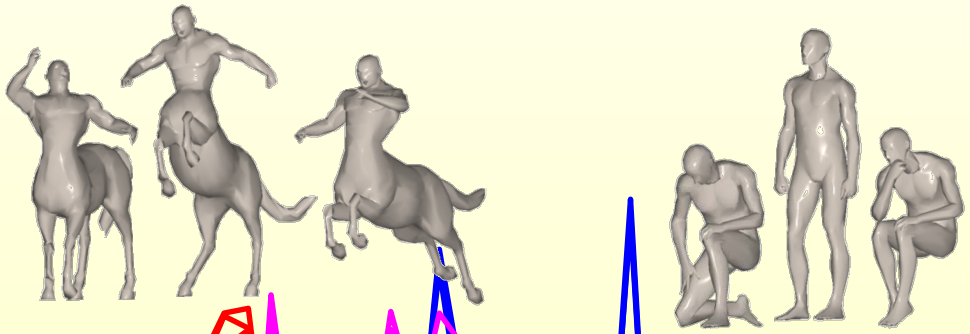
ShapeGoogle: HKS-Based BoW Shape Search

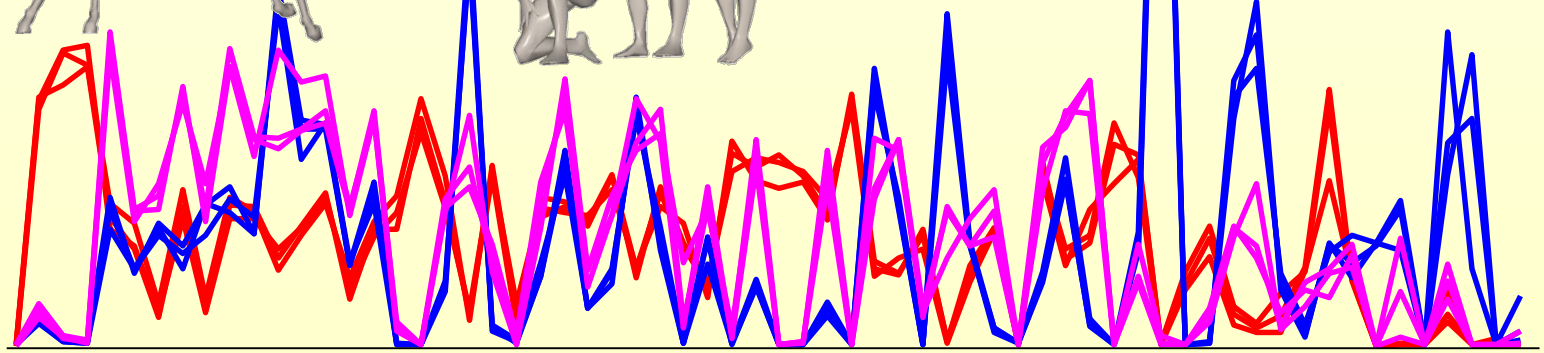


Shape Signatures



Bags of features





1

Index in vocabulary

The End